CHAPTER 3

More on order theory

3.1. Straight maps and separation subsets

3.1.1. Straight maps.

DEFINITION 204. An *order reflecting* map from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ is such a function f that (for every $x, y \in \mathfrak{A}$)

$$fx \sqsubseteq fy \Rightarrow x \sqsubseteq y.$$

OBVIOUS 205. Order embeddings are exactly the same as monotone and order reflecting maps.

DEFINITION 206. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. I call f a *straight* map when

$$\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa = f(a \sqcap b)).$$

PROPOSITION 207. The following statements are equivalent for a monotone map f:

1°. f is a straight map.
2°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa \sqsubseteq f(a \sqcap b))$.
3°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa \not\sqsupset f(a \sqcap b))$.
4°. $\forall a, b \in \mathfrak{A} : (fa \sqsupset f(a \sqcap b) \Rightarrow fa \not\sqsubseteq fb)$.

PROOF.
1°⇔2°⇔3°. Due $fa \sqsupseteq f(a \sqcap b)$.
3°⇔4°. Obvious.

□

REMARK 208. The definition of straight map can be generalized for any poset $\mathfrak{A}$ by the formula

$$\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow \exists c \in \mathfrak{A} : (c \sqsubseteq a \wedge c \sqsubseteq b \wedge fa = fc)).$$

This generalization is not yet researched however.

PROPOSITION 209. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to a meet-semilattice $\mathfrak{B}$. If

$$\forall a, b \in \mathfrak{A} : f(a \sqcap b) = fa \sqcap fb$$

then f is a straight map.

PROOF. Let $fa \sqsubseteq fb$. Then $f(a \sqcap b) = fa \sqcap fb = fa$. □

PROPOSITION 210. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. If f is order reflecting, then f is a straight map.

PROOF. $fa \sqsubseteq fb \Rightarrow a \sqsubseteq b \Rightarrow a = a \sqcap b \Rightarrow fa = f(a \sqcap b)$. □

The following theorem is the main reason of why we are interested in straight maps: