PROOF. It is enough to prove $f \sqsubseteq g \Rightarrow f_* \sqsupseteq g_*$ (the rest follows from the fact that a Galois connection is determined by one adjoint).

Really, let $f \sqsubseteq g$. Then $f_0^* \sqsubseteq f_1^*$ and thus:
$f_{0*}(b) = \max\left\{\frac{x\in\mathfrak{A}}{f_0^* x \sqsubseteq b}\right\}$, $f_{1*}(b) = \max\left\{\frac{x\in\mathfrak{A}}{f_1^* x \sqsubseteq b}\right\}$.
Thus $f_{0*}(b) \sqsupseteq f_{1*}(b)$ for every $b \in \mathfrak{B}$ and so $f_{0*} \sqsupseteq f_{1*}$. □

DEFINITION 137. Composition of Galois connections is defined by the formula: $g \circ f = (g^* \circ f^*, f_* \circ g_*)$.

PROPOSITION 138. Composition of Galois connections is a Galois connection.

PROOF. $g^* \circ f^*$ and $f_* \circ g_*$ are monotone as composition of monotone functions;

$$(g^* \circ f^*)x \sqsubseteq z \Leftrightarrow g^* f^* x \sqsubseteq z \Leftrightarrow f^* x \sqsubseteq g_* z \Leftrightarrow x \sqsubseteq f_* g_* z \Leftrightarrow x \sqsubseteq (f_* \circ g_*)z.$$

□

OBVIOUS 139. Composition of Galois connections preserves order.

2.1.13.2. *Antitone Galois connections.*

DEFINITION 140. An *antitone Galois connection* between posets $\mathfrak{A}$ and $\mathfrak{B}$ is a Galois connection between $\mathfrak{A}$ and $\operatorname{dual}\mathfrak{B}$.

OBVIOUS 141. An antitone Galois connection is a pair of antitone functions $f : \mathfrak{A} \to \mathfrak{B}$, $g : \mathfrak{B} \to \mathfrak{A}$ such that $b \sqsubseteq fa \Leftrightarrow a \sqsubseteq gb$ for every $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.

Such f and g are called *polarities* (between $\mathfrak{A}$ and $\mathfrak{B}$).

OBVIOUS 142. $f \bigsqcup S = \bigsqcap \langle f \rangle^* S$ if f is a polarity between $\mathfrak{A}$ and $\mathfrak{B}$ and $S \in \mathscr{P}\mathfrak{A}$.

Galois connections (particularly between boolean lattices) are studied in [32] and [33].

2.1.14. Co-Brouwerian lattices.

DEFINITION 143. Let $\mathfrak{A}$ be a poset. *Pseudocomplement* of $a \in \mathfrak{A}$ is

$$\max\left\{\frac{c \in \mathfrak{A}}{c \asymp a}\right\}.$$

If z is the pseudocomplement of a we will denote $z = a^*$.

DEFINITION 144. Let $\mathfrak{A}$ be a poset. *Dual pseudocomplement* of $a \in \mathfrak{A}$ is

$$\min\left\{\frac{c \in \mathfrak{A}}{c \equiv a}\right\}.$$

If z is the dual pseudocomplement of a we will denote $z = a^+$.

PROPOSITION 145. If a is a complemented element of a bounded distributive lattice, then $\bar{a}$ is both pseudocomplement and dual pseudocomplement of a.

PROOF. Because of duality it is enough to prove that $\bar{a}$ is pseudocomplement of a.

We need to prove $c \asymp a \Rightarrow c \sqsubseteq \bar{a}$ for every element c of our poset, and $\bar{a} \asymp a$. The second is obvious. Let's prove $c \asymp a \Rightarrow c \sqsubseteq \bar{a}$.

Really, let $c \asymp a$. Then $c \sqcap a = \bot$; $\bar{a} \sqcup (c \sqcap a) = \bar{a}$; $(\bar{a} \sqcup c) \sqcap (\bar{a} \sqcup a) = \bar{a}$; $\bar{a} \sqcup c = \bar{a}$; $c \sqsubseteq \bar{a}$. □

DEFINITION 146. Let $\mathfrak{A}$ be a join-semilattice. Let $a, b \in \mathfrak{A}$. *Pseudodifference* of a and b is

$$\min\left\{\frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z}\right\}.$$

If z is a pseudodifference of a and b we will denote $z = a \setminus^* b$.