1°. f^* and f_* are monotone.
2°. $x \sqsubseteq f_* f^* x$ and $f^* f_* y \sqsubseteq y$ for every $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$.

Proof.

$\Rightarrow$.

2°. $x \sqsubseteq f_* f^* x$ since $f^* x \sqsubseteq f^* x$; $f^* f_* y \sqsubseteq y$ since $f_* y \sqsubseteq f_* y$.
1°. Let $a, b \in \mathfrak{A}$ and $a \sqsubseteq b$. Then $a \sqsubseteq b \sqsubseteq f_* f^* b$. So by definition $f^* a \sqsubseteq f^* b$ that is f^* is monotone. Analogously f_* is monotone.

$\Leftarrow$. $f^* x \sqsubseteq y \Rightarrow f_* f^* x \sqsubseteq f_* y \Rightarrow x \sqsubseteq f_* y$. The other direction is analogous.

□

Theorem 127.
1°. $f^* \circ f_* \circ f^* = f^*$.
2°. $f_* \circ f^* \circ f_* = f_*$.

Proof.
1°. Let $x \in \mathfrak{A}$. We have $x \sqsubseteq f_* f^* x$; consequently $f^* x \sqsubseteq f^* f_* f^* x$. On the other hand, $f^* f_* f^* x \sqsubseteq f^* x$. So $f^* f_* f^* x = f^* x$.
2°. Similar.

□

Definition 128. A function f is called *idempotent* iff $f(f(X)) = f(X)$ for every argument X.

Proposition 129. $f^* \circ f_*$ and $f_* \circ f^*$ are idempotent.

Proof. $f^* \circ f_*$ is idempotent because $f^* f_* f^* f_* y = f^* f_* y$. $f_* \circ f^*$ is similar. □

Theorem 130. Each of two adjoints is uniquely determined by the other.

Proof. Let p and q be both upper adjoints of f. We have for all $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$:

$$x \sqsubseteq p(y) \Leftrightarrow f(x) \sqsubseteq y \Leftrightarrow x \sqsubseteq q(y).$$

For $x = p(y)$ we obtain $p(y) \sqsubseteq q(y)$ and for $x = q(y)$ we obtain $q(y) \sqsubseteq p(y)$. So $q(y) = p(y)$. □

Theorem 131. Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.
1°. Both:
(a) If f is monotone and $g(b) = \max\left\{ \frac{x \in \mathfrak{A}}{fx \sqsubseteq b} \right\}$ is defined for every $b \in \mathfrak{B}$ then g is the upper adjoint of f.
(b) If $g : \mathfrak{B} \to \mathfrak{A}$ is the upper adjoint of f then $g(b) = \max\left\{ \frac{x \in \mathfrak{A}}{fx \sqsubseteq b} \right\}$ for every $b \in \mathfrak{B}$.
2°. Both:
(a) If f is monotone and $g(b) = \min\left\{ \frac{x \in \mathfrak{A}}{fx \sqsupseteq b} \right\}$ is defined for every $b \in \mathfrak{B}$ then g is the lower adjoint of f.
(b) If $g : \mathfrak{B} \to \mathfrak{A}$ is the lower adjoint of f then $g(b) = \min\left\{ \frac{x \in \mathfrak{A}}{fx \sqsupseteq b} \right\}$ for every $b \in \mathfrak{B}$.

Proof. We will prove only the first as the second is its dual.

1°a. Let $g(b) = \max\left\{ \frac{x \in \mathfrak{A}}{fx \sqsubseteq b} \right\}$ for every $b \in \mathfrak{B}$. Then

$$x \sqsubseteq gy \Leftrightarrow x \sqsubseteq \max\left\{ \frac{x \in \mathfrak{A}}{fx \sqsubseteq y} \right\} \Rightarrow fx \sqsubseteq y$$