DEFINITION 104. *Atomistic poset* is such a poset that $a = \bigsqcup \operatorname{atoms} a$ for every element a of this poset.

OBVIOUS 105. Every atomistic poset is atomic.

PROPOSITION 106. Let $\mathfrak{A}$ be a poset. If a is an atom of $\mathfrak{A}$ and $B \in \mathfrak{A}$ then

$$a \in \operatorname{atoms} B \Leftrightarrow a \sqsubseteq B \Leftrightarrow a \not\asymp B.$$

PROOF.

$a \in \operatorname{atoms} B \Leftrightarrow a \sqsubseteq B$. Obvious.

$a \sqsubseteq B \Rightarrow a \not\asymp B$. $a \sqsubseteq B \Rightarrow a \sqsubseteq a \wedge a \sqsubseteq B$, thus $a \not\asymp B$ because a is not least.

$a \sqsubseteq B \Leftarrow a \not\asymp B$. $a \not\asymp B$ implies existence of non-least element x such that $x \sqsubseteq B$ and $x \sqsubseteq a$. Because a is an atom, we have $x = a$. So $a \sqsubseteq B$.

□

THEOREM 107. A poset is atomistic iff every its element can be represented as join of atoms.

PROOF.

$\Rightarrow$. Obvious.

$\Leftarrow$. Let $a = \bigsqcup S$ where S is a set of atoms. We will prove that a is the least upper bound of $\operatorname{atoms} a$.

That a is an upper bound of $\operatorname{atoms} a$ is obvious. Let x is an upper bound of $\operatorname{atoms} a$. Then $x \sqsupseteq \bigsqcup S$ because $S \subseteq \operatorname{atoms} a$. Thus $x \sqsupseteq a$.

□

THEOREM 108. $\operatorname{atoms} \bigsqcap S = \bigcap \langle \operatorname{atoms} \rangle^* S$ whenever $\bigsqcap S$ is defined for every $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset.

PROOF. For any atom

$$\begin{aligned} c \in \operatorname{atoms} \bigsqcap S &\Leftrightarrow \\ c \sqsubseteq \bigsqcap S &\Leftrightarrow \\ \forall a \in S : c \sqsubseteq a &\Leftrightarrow \\ \forall a \in S : c \in \operatorname{atoms} a &\Leftrightarrow \\ c \in \bigcap \langle \operatorname{atoms} \rangle^* S&. \end{aligned}$$

□

COROLLARY 109. $\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ for an arbitrary meet-semilattice.

THEOREM 110. A complete boolean lattice is atomic iff it is atomistic.

PROOF.

$\Leftarrow$. Obvious.

$\Rightarrow$. Let $\mathfrak{A}$ be an atomic boolean lattice. Let $a \in \mathfrak{A}$. Suppose $b = \bigsqcup \operatorname{atoms} a \sqsubset a$. If $x \in \operatorname{atoms}(a \setminus b)$ then $x \sqsubseteq a \setminus b$ and so $x \sqsubseteq a$ and hence $x \sqsubseteq b$. But we have $x = x \sqcap b \sqsubseteq (a \setminus b) \sqcap b = \bot$ what contradicts to our supposition.

□

2.1.11. Kuratowski's lemma.

THEOREM 111. (KURATOWSKI's lemma) Any chain in a poset is contained in a maximal chain (if we order chains by inclusion).

I will skip the proof of KURATOWSKI's lemma as this proof can be found in any set theory or order theory reference.