2.1.9. Center of a lattice.

DEFINITION 91. The *center* $Z(\mathfrak{A})$ of a bounded distributive lattice $\mathfrak{A}$ is the set of its complemented elements.

REMARK 92. For a definition of center of non-distributive lattices see [5].

REMARK 93. In [24] the word center and the notation $Z(\mathfrak{A})$ are used in a different sense.

DEFINITION 94. A sublattice K of a complete lattice L is a *closed sublattice* of L if K contains the meet and the join of any its nonempty subset.

THEOREM 95. Center of an infinitely distributive lattice is its closed sublattice.

PROOF. See [17]. □

REMARK 96. See [18] for a more strong result.

THEOREM 97. The center of a bounded distributive lattice constitutes its sublattice.

PROOF. Let $\mathfrak{A}$ be a bounded distributive lattice and $Z(\mathfrak{A})$ be its center. Let $a, b \in Z(\mathfrak{A})$. Consequently $\bar{a}, \bar{b} \in Z(\mathfrak{A})$. Then $\bar{a} \sqcup \bar{b}$ is the complement of $a \sqcap b$ because

$$(a \sqcap b) \sqcap (\bar{a} \sqcup \bar{b}) = (a \sqcap b \sqcap \bar{a}) \sqcup (a \sqcap b \sqcap \bar{b}) = \bot \sqcup \bot = \bot \quad \text{and}$$
$$(a \sqcap b) \sqcup (\bar{a} \sqcup \bar{b}) = (a \sqcup \bar{a} \sqcup \bar{b}) \sqcap (b \sqcup \bar{a} \sqcup \bar{b}) = \top \sqcap \top = \top.$$

So $a \sqcap b$ is complemented. Similarly $a \sqcup b$ is complemented. □

THEOREM 98. The center of a bounded distributive lattice constitutes a boolean lattice.

PROOF. Because it is a distributive complemented lattice. □

2.1.10. Atoms of posets.

DEFINITION 99. An atom of a poset is an element a such that (for every its element x) $x \sqsubset a$ if and only if x is the least element.

REMARK 100. This definition is valid even for posets without least element.

PROPOSITION 101. Element a is an atom iff both:

1°. $x \sqsubset a$ implies x is the least element;
2°. a is non-least.

PROOF.

⇒. Let a be an atom. 1° is obvious. If a is least then $a \sqsubset a$ what is impossible, so 2°.

⇐. Let 1° and 2° hold. We need to prove only that x is least implies that $x \sqsubset a$ but this follows from a being non-least.

□

EXAMPLE 102. Atoms of the boolean algebra $\mathscr{P}A$ (ordered by set inclusion) are one-element sets.

I will denote $\operatorname{atoms}^{\mathfrak{A}} a$ or just $(\operatorname{atoms} a)$ the set of atoms contained in an element a of a poset $\mathfrak{A}$. I will denote $\operatorname{atoms}^{\mathfrak{A}}$ the set of all atoms of a poset $\mathfrak{A}$.

DEFINITION 103. A poset $\mathfrak{A}$ is called *atomic* iff $\operatorname{atoms} a \neq \emptyset$ for every non-least element a of the poset $\mathfrak{A}$.