2.1.8. Boolean lattices.

DEFINITION 84. A *boolean lattice* is a complemented distributive lattice.

The most important example of a boolean lattice is $\mathscr{P}A$ where A is a set, ordered by set inclusion.

THEOREM 85. (DE MORGAN's laws) For every elements a, b of a boolean lattice

1°. $\overline{a \sqcup b} = \bar{a} \sqcap \bar{b}$;
2°. $\overline{a \sqcap b} = \bar{a} \sqcup \bar{b}$.

PROOF. We will prove only the first as the second is dual.

It is enough to prove that $a \sqcup b$ is a complement of $\bar{a} \sqcap \bar{b}$. Really:

$$(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) \sqsubseteq a \sqcap (\bar{a} \sqcap \bar{b}) = (a \sqcap \bar{a}) \sqcap \bar{b} = \bot \sqcap \bar{b} = \bot;$$

$$(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = ((a \sqcup b) \sqcup \bar{a}) \sqcap ((a \sqcup b) \sqcup \bar{b}) \sqsupseteq (a \sqcup \bar{a}) \sqcap (b \sqcup \bar{b}) = \top \sqcap \top = \top.$$

Thus $(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) = \bot$ and $(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = \top$. □

DEFINITION 86. A complete lattice $\mathfrak{A}$ is *join infinite distributive* when $x \sqcap \bigsqcup S = \bigsqcup \langle x \sqcap \rangle^* S$; a complete lattice $\mathfrak{A}$ is *meet infinite distributive* when $x \sqcup \bigsqcap S = \bigsqcap \langle x \sqcup \rangle^* S$ for all $x \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$.

DEFINITION 87. *Infinite distributive complete lattice* is a complete lattice which is both join infinite distributive and meet infinite distributive.

THEOREM 88. For every boolean lattice $\mathfrak{A}$, $x \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$ we have:

1°. $\bigsqcup \langle x \sqcap \rangle^* S$ is defined and $x \sqcap \bigsqcup S = \bigsqcup \langle x \sqcap \rangle^* S$ whenever $\bigsqcup S$ is defined.
2°. $\bigsqcap \langle x \sqcup \rangle^* S$ is defined and $x \sqcup \bigsqcap S = \bigsqcap \langle x \sqcup \rangle^* S$ whenever $\bigsqcap S$ is defined.

PROOF. We will prove only the first, as the other is dual.

We need to prove that $x \sqcap \bigsqcup S$ is the least upper bound of $\langle x \sqcap \rangle^* S$.

That $x \sqcap \bigsqcup S$ is an upper bound of $\langle x \sqcap \rangle^* S$ is obvious.

Now let u be any upper bound of $\langle x \sqcap \rangle^* S$, that is $x \sqcap y \sqsubseteq u$ for all $y \in S$. Then

$$y = y \sqcap (x \sqcup \bar{x}) = (y \sqcap x) \sqcup (y \sqcap \bar{x}) \sqsubseteq u \sqcup \bar{x},$$

and so $\bigsqcup S \sqsubseteq u \sqcup \bar{x}$. Thus

$$x \sqcap \bigsqcup S \sqsubseteq x \sqcap (u \sqcup \bar{x}) = (x \sqcap u) \sqcup (x \sqcap \bar{x}) = (x \sqcap u) \sqcup \bot = x \sqcap u \sqsubseteq u,$$

that is $x \sqcap \bigsqcup S$ is the least upper bound of $\langle x \sqcap \rangle^* S$. □

COROLLARY 89. Every complete boolean lattice is both join infinite distributive and meet infinite distributive.

THEOREM 90. (infinite DE MORGAN's laws) For every subset S of a complete boolean lattice

1°. $\overline{\bigsqcup S} = \bigsqcap_{x \in S} \bar{x}$;
2°. $\overline{\bigsqcap S} = \bigsqcup_{x \in S} \bar{x}$.

PROOF. It's enough to prove that $\bigsqcup S$ is a complement of $\bigsqcap_{x \in S} \bar{x}$ (the second follows from duality). Really, using the previous theorem:

$$\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = \bigsqcap_{x \in S} \left\langle \bigsqcup S \sqcup \right\rangle^* \bar{x} = \bigsqcap \left\{ \frac{\bigsqcup S \sqcup \bar{x}}{x \in S} \right\} \sqsupseteq \bigsqcap \left\{ \frac{x \sqcup \bar{x}}{x \in S} \right\} = \top;$$

$$\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = \bigsqcup_{y \in S} \left\langle \bigsqcap_{x \in S} \bar{x} \sqcap \right\rangle^* y = \bigsqcup \left\{ \frac{\bigsqcap_{x \in S} \bar{x} \sqcap y}{y \in S} \right\} \sqsubseteq \bigsqcup \left\{ \frac{\bar{y} \sqcap y}{y \in S} \right\} = \bot.$$

So $\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = \top$ and $\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = \bot$. □