1°. $\bigsqcup \bigcup S = \bigsqcup_{X \in S} \bigsqcup X$;
2°. $\bigsqcap \bigcup S = \bigsqcap_{X \in S} \bigsqcap X$.

Proof. We will prove only the first as the second is dual.
By definition of joins, it is enough to prove $y \sqsupseteq \bigsqcup \bigcup S \Leftrightarrow y \sqsupseteq \bigsqcup_{X \in S} \bigsqcup X$.
Really,

$$\begin{aligned} y \sqsupseteq \bigsqcup \bigcup S \Leftrightarrow \\ \forall x \in \bigcup S : y \sqsupseteq x \Leftrightarrow \\ \forall X \in S \forall x \in X : y \sqsupseteq x \Leftrightarrow \\ \forall X \in S : y \sqsupseteq \bigsqcup X \Leftrightarrow \\ y \sqsupseteq \bigsqcup_{X \in S} \bigsqcup X. \end{aligned}$$

□

Definition 69. A *sublattice* of a lattice is it subset closed regarding $\sqcup$ and $\sqcap$.

Obvious 70. Sublattice with induced order is also a lattice.

2.1.6. Distributivity of lattices.

Definition 71. A *distributive* lattice is such lattice $\mathfrak{A}$ that for every $x, y, z \in \mathfrak{A}$

1°. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2°. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

Theorem 72. For a lattice to be distributive it is enough just one of the conditions:

1°. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2°. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

Proof.

$$\begin{aligned} (x \sqcup y) \sqcap (x \sqcup z) = \\ ((x \sqcup y) \sqcap x) \sqcup ((x \sqcup y) \sqcap z) = \\ x \sqcup ((x \sqcap z) \sqcup (y \sqcap z)) = \\ (x \sqcup (x \sqcap z)) \sqcup (y \sqcap z) = \\ x \sqcup (y \sqcap z) \end{aligned}$$

(applied $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ twice). □

2.1.7. Difference and complement.

Definition 73. Let $\mathfrak{A}$ be a distributive lattice with least element $\bot$. The *difference* (denoted $a \setminus b$) of elements a and b is such $c \in \mathfrak{A}$ that $b \sqcap c = \bot$ and $a \sqcup b = b \sqcup c$. I will call b *substractive* from a when $a \setminus b$ exists.

Theorem 74. If $\mathfrak{A}$ is a distributive lattice with least element $\bot$, there exists no more than one difference of elements a, b.

Proof. Let c and d be both differences $a \setminus b$. Then $b \sqcap c = b \sqcap d = \bot$ and $a \sqcup b = b \sqcup c = b \sqcup d$. So

$$c = c \sqcap (b \sqcup c) = c \sqcap (b \sqcup d) = (c \sqcap b) \sqcup (c \sqcap d) = \bot \sqcup (c \sqcap d) = c \sqcap d.$$

Similarly $d = d \sqcap c$. Consequently $c = c \sqcap d = d \sqcap c = d$. □