

### 2.1.4. Semilattices.

DEFINITION 57.

- 1°. A *join-semilattice* is a poset  $\mathfrak{A}$  such that  $a \sqcup b$  is defined for every  $a, b \in \mathfrak{A}$ .
- 2°. A *meet-semilattice* is a poset  $\mathfrak{A}$  such that  $a \sqcap b$  is defined for every  $a, b \in \mathfrak{A}$ .

THEOREM 58.

- 1°. The operation  $\sqcup$  is associative for any join-semilattice.
- 2°. The operation  $\sqcap$  is associative for any meet-semilattice.

PROOF. I will prove only the first as the second follows by duality.

We need to prove  $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$  for every  $a, b, c \in \mathfrak{A}$ .

Taking into account the definition of join, it is enough to prove that

$$x \sqsupseteq (a \sqcup b) \sqcup c \Leftrightarrow x \sqsupseteq a \sqcup (b \sqcup c)$$

for every  $x \in \mathfrak{A}$ . Really, this follows from the chain of equivalences:

$$\begin{aligned} x \sqsupseteq (a \sqcup b) \sqcup c &\Leftrightarrow \\ x \sqsupseteq a \sqcup b \wedge x \sqsupseteq c &\Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \wedge x \sqsupseteq c &\Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \sqcup c &\Leftrightarrow \\ x \sqsupseteq a \sqcup (b \sqcup c). & \end{aligned}$$

□

OBVIOUS 59.  $a \not\leq b$  iff  $a \sqcap b$  is non-least, for every elements  $a, b$  of a meet-semilattice.

OBVIOUS 60.  $a \equiv b$  iff  $a \sqcup b$  is the greatest element, for every elements  $a, b$  of a join-semilattice.

### 2.1.5. Lattices and complete lattices.

DEFINITION 61. A *bounded* poset is a poset having both least and greatest elements.

DEFINITION 62. *Lattice* is a poset which is both join-semilattice and meet-semilattice.

DEFINITION 63. A *complete lattice* is a poset  $\mathfrak{A}$  such that for every  $X \in \mathcal{P}\mathfrak{A}$  both  $\bigsqcup X$  and  $\bigsqcap X$  exist.

OBVIOUS 64. Every complete lattice is a lattice.

PROPOSITION 65. Every complete lattice is a bounded poset.

PROOF.  $\bigsqcup \emptyset$  is the least and  $\bigsqcap \emptyset$  is the greatest element. □

THEOREM 66. Let  $\mathfrak{A}$  be a poset.

- 1°. If  $\bigsqcup X$  is defined for every  $X \in \mathcal{P}\mathfrak{A}$ , then  $\mathfrak{A}$  is a complete lattice.
- 2°. If  $\bigsqcap X$  is defined for every  $X \in \mathcal{P}\mathfrak{A}$ , then  $\mathfrak{A}$  is a complete lattice.

PROOF. See [27] or any lattice theory reference. □

OBVIOUS 67. If  $X \subseteq Y$  for some  $X, Y \in \mathcal{P}\mathfrak{A}$  where  $\mathfrak{A}$  is a complete lattice, then

- 1°.  $\bigsqcup X \subseteq \bigsqcup Y$ ;
- 2°.  $\bigsqcap X \supseteq \bigsqcap Y$ .

PROPOSITION 68. If  $S \in \mathcal{P}\mathcal{P}\mathfrak{A}$  then for every complete lattice  $\mathfrak{A}$