REMARK 42. Least and greatest elements of a set X is a trivial generalization of the above defined least and greatest element for the entire poset.

DEFINITION 43.

- A *minimal* element of a set $X \in \mathscr{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X : a \sqsupset x$.
- A *maximal* element of a set $X \in \mathscr{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X : a \sqsubset x$.

REMARK 44. Minimal element is not the same as minimum, and maximal element is not the same as maximum.

OBVIOUS 45.

1°. The least element (if it exists) is a minimal element.
2°. The greatest element (if it exists) is a maximal element.

EXERCISE 46. Show that there may be more than one minimal and more than one maximal element for some poset.

DEFINITION 47. *Upper bounds* of a set X is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsupseteq x} \right\}$.

The dual notion:

DEFINITION 48. *Lower bounds* of a set X is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsubseteq x} \right\}$.

DEFINITION 49. *Join* $\bigsqcup X$ (also called *supremum* and denoted "$\sup X$") of a set X is the least element of its upper bounds (if it exists).

DEFINITION 50. *Meet* $\bigsqcap X$ (also called *infimum* and denoted "$\inf X$") of a set X is the greatest element of its lower bounds (if it exists).

We will also denote $\bigsqcup_{i \in X} f(i) = \bigsqcup \left\{ \frac{f(i)}{x \in X} \right\}$ and $\bigsqcap_{i \in X} f(i) = \bigsqcap \left\{ \frac{f(i)}{x \in X} \right\}$.

We will write $b = \bigsqcup X$ when $b \in \mathfrak{A}$ is the join of X or say that $\bigsqcup X$ does not exist if there are no such $b \in \mathfrak{A}$. (And dually for meets.)

EXERCISE 51. Provide an example of $\bigsqcup X \notin X$ for some set X on some poset.

PROPOSITION 52.

1°. If b is the greatest element of X then $\bigsqcup X = b$.
2°. If b is the least element of X then $\bigsqcap X = b$.

PROOF. We will prove only the first as the second is dual.

Let b be the greatest element of X. Then upper bounds of X are $\left\{ \frac{y \in \mathfrak{A}}{y \sqsupseteq b} \right\}$. Obviously b is the least element of this set, that is the join. □

DEFINITION 53. *Binary joins and meets* are defined by the formulas

$$x \sqcup y = \bigsqcup \{x, y\} \quad \text{and} \quad x \sqcup y = \bigsqcap \{x, y\}.$$

OBVIOUS 54. $\sqcup$ and $\sqcap$ are symmetric operations (whenever these are defined for given x and y).

THEOREM 55.

1°. If $\bigsqcup X$ exists then $y \sqsupseteq \bigsqcup X \Leftrightarrow \forall x \in X : y \sqsupseteq x$.
2°. If $\bigsqcap X$ exists then $y \sqsubseteq \bigsqcap X \Leftrightarrow \forall x \in X : y \sqsubseteq x$.

PROOF. I will prove only the first as the second follows by duality.
$y \sqsupseteq \bigsqcup X \Leftrightarrow y$ is an upper bound for $X \Leftrightarrow \forall x \in X : y \sqsupseteq x$. □

COROLLARY 56.

1°. If $a \sqcup b$ exists then $y \sqsupseteq a \sqcup b \Leftrightarrow y \sqsupseteq a \land y \sqsupseteq b$.
2°. If $a \sqcap b$ exists then $y \sqsubseteq a \sqcap b \Leftrightarrow y \sqsubseteq a \land y \sqsubseteq b$.

I will denote meets and joins for a specific poset $\mathfrak{A}$ as $\bigsqcap^{\mathfrak{A}}$, $\bigsqcup^{\mathfrak{A}}$, $\sqcap^{\mathfrak{A}}$, $\sqcup^{\mathfrak{A}}$.