CHAPTER 2

Common knowledge, part 1

In this chapter we will consider some well known mathematical theories. If you already know them you may skip reading this chapter (or its parts).

2.1. Order theory

2.1.1. Posets.

DEFINITION 14. The *identity relation* on a set A is $\mathrm{id}_A = \left\{ \frac{(a,a)}{a \in A} \right\}$.

DEFINITION 15. A *preorder* on a set A is a binary relation $\sqsubseteq$ on A which is:

- *reflexive* on A that is $(\sqsubseteq) \supseteq \mathrm{id}_A$ or what is the same $\forall x \in A : x \sqsubseteq x$;
- *transitive* that is $(\sqsubseteq) \circ (\sqsubseteq) \subseteq (\sqsubseteq)$ or what is the same

$$\forall x, y, z : (x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z).$$

DEFINITION 16. A *partial order* on a set A is a preorder on A which is *antisymmetric* that is $(\sqsubseteq) \cap (\sqsupseteq) \subseteq \mathrm{id}_A$ or what is the same

$$\forall x, y \in A : (x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y).$$

The reverse relation is denoted $\sqsupseteq$.

DEFINITION 17. a is a subelement of b (or what is the same a is *contained* in b or b *contains* a) iff $a \sqsubseteq b$.

OBVIOUS 18. The reverse of a partial order is also a partial order.

DEFINITION 19. A set A together with a partial order on it is called a *partially ordered set* (*poset* for short).

An example of a poset is the set $\mathbb{R}$ of real numbers with $\sqsubseteq = \leq$.

Another example is the set $\mathscr{P}A$ of all subsets of an arbitrary fixed set A with $\sqsubseteq = \subseteq$. Note that this poset is (in general) not linear (see definition of *linear* poset below.)

DEFINITION 20. Strict partial order $\sqsubset$ corresponding to the partial order $\sqsubseteq$ on a set A is defined by the formula $(\sqsubset) = (\sqsubseteq) \setminus \mathrm{id}_A$. In other words,

$$a \sqsubset b \Leftrightarrow a \sqsubseteq b \wedge a \neq b.$$

An example of strict partial order is $<$ on the set $\mathbb{R}$ of real numbers.

DEFINITION 21. A partial order on a set A *restricted* to a set $B \subseteq A$ is $(\sqsubseteq) \cap (B \times B)$.

OBVIOUS 22. A partial order on a set A restricted to a set $B \subseteq A$ is a partial order on B.

DEFINITION 23.

- The *least* element $\bot$ of a poset $\mathfrak{A}$ is defined by the formula $\forall a \in \mathfrak{A} : \bot \sqsubseteq a$.
- The *greatest* element $\top$ of a poset $\mathfrak{A}$ is defined by the formula $\forall a \in \mathfrak{A} : \top \sqsupseteq a$.