Proof. $\forall A\in a\colon \prod X\not\asymp \mathrm{id}_{A[n]}\Leftrightarrow \forall A\in a\colon \bigcap_{i\in n} X_i\cap A\neq\emptyset \Leftrightarrow \forall A\in a\colon \prod^{\mathfrak{P}}_{i\in n}\uparrow^3 X_i\not\asymp\uparrow A\Leftrightarrow \prod^{\mathfrak{P}}_{i\in n}(\uparrow^{3^n}X_i)\not\asymp a\Leftrightarrow \prod^{\mathfrak{P}}_{i\in n}(\uparrow^{3^n}X)_i\not\asymp a\Leftrightarrow\uparrow^{3^n}X\in\mathrm{GR}\,\mathrm{id}_{a[n]}$. □

Proposition 18.51. $Y\in\mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow\forall A\in\mathcal{A}\colon Y\in\mathrm{GR}\uparrow^{\mathsf{Strd}}\mathrm{id}_{A[n]}$ for every filter $\mathcal{A}$ on a powerset and $Y\in\mathfrak{P}^n$.

Proof. Take $Y=\uparrow^{3^n}X$.

$\forall A\in\mathcal{A}\colon Y\in\mathrm{GR}\uparrow^{\mathsf{Strd}}\mathrm{id}_{A[n]}\Leftrightarrow\forall A\in\mathcal{A}\colon\uparrow^{3^n}X\in\mathrm{GR}\uparrow^{\mathsf{Strd}}\mathrm{id}_{A[n]}\Leftrightarrow\forall A\in\mathcal{A}\colon\prod X\not\asymp\mathrm{id}_{A[n]}\Leftrightarrow\uparrow^{3^n}X\in\mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow Y\in\mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. □

Proposition 18.52. $\uparrow^{3^n}X\in\mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[n]}\Leftrightarrow\forall A\in a\exists t\in A\forall i\in n\colon t\in X_i$.

Proof. $\uparrow^{3^n}X\in\mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[n]}\Leftrightarrow\exists A\in a\exists t\in A\colon n\times\{t\}\in\prod X\Leftrightarrow\forall A\in a\exists t\in A\forall i\in n\colon t\in X_i$. □

18.4.5 Relationships between big and small identity staroids

Definition 18.53. $a^n_{\mathsf{Strd}}=\prod^{\mathsf{Strd}}_{i\in n}a$ for every element a of a poset and an index set n.

Proposition 18.54. $\upuparrows\mathrm{id}^{\mathsf{Strd}}_{a[n]}\sqsubseteq\mathrm{ID}^{\mathsf{Strd}}_{a[n]}\sqsubseteq a^n_{\mathsf{Strd}}$ for every filter a (on any distributive lattice) and an index set n.

Proof.

$\mathrm{GR}\upuparrows\mathrm{id}^{\mathsf{Strd}}_{a[n]}\subseteq\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$. $\mathcal{L}\in\mathrm{GR}\upuparrows\mathrm{id}^{\mathsf{Strd}}_{a[n]}\Leftrightarrow\mathrm{up}\,\mathcal{L}\subseteq\mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[n]}\Leftrightarrow\forall L\in\mathrm{up}\,\mathcal{L}\colon L\in\mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[n]}\Leftrightarrow$ (proposition 4.112)$\Leftrightarrow\forall L\in\mathrm{up}\,\mathcal{L}\forall A\in\mathrm{up}\,a\colon\prod^3_{i\in n}L_i\not\asymp A\Leftrightarrow\forall L\in\mathrm{up}\,\mathcal{L}\forall A\in\mathrm{up}\,a\colon\prod^3_{i\in n}L_i\sqcap A\neq 0\Rightarrow\bigcup_{i\in n}\mathcal{L}_i\cup a$ has finite intersection property$\Leftrightarrow\mathcal{L}\in\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$.

$\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}\subseteq\mathrm{GR}\,a^n_{\mathsf{Strd}}$. $\mathcal{L}\in\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}\Leftrightarrow\mathrm{MEET}(\{\mathcal{L}_i\mid i\in n\}\cup\{a\})\Rightarrow\forall i\in a\colon\mathcal{L}_i\not\asymp a\Leftrightarrow\mathcal{L}\in\mathrm{GR}\,a^a_{\mathsf{Strd}}$. □

Proposition 18.55. $\upuparrows\mathrm{id}^{\mathsf{Strd}}_{a[a]}\sqsubset\mathrm{ID}^{\mathsf{Strd}}_{a[a]}=a^a_{\mathsf{Strd}}$ for every nontrivial ultrafilter a on a set.

Proof.

$\mathrm{GR}\upuparrows\mathrm{id}^{\mathsf{Strd}}_{a[a]}\neq\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[a]}$. Let $\mathcal{L}_i=\uparrow^{\mathrm{Base}(a)}i$. Then trivially $\mathcal{L}\in\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[a]}$. But to disprove $\mathcal{L}\in\mathrm{GR}\upuparrows\mathrm{id}^{\mathsf{Strd}}_{a[a]}$ it's enough to show $L\notin\mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[a]}$ for some $L\in\mathrm{up}\,\mathcal{L}$. Really, take $L_i=\mathcal{L}_i=\uparrow^{\mathrm{Base}(a)}i$. Then $L\in\mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[a]}\Leftrightarrow\forall A\in a\exists t\in A\forall i\in a\colon t\in i$ what is clearly false (we can always take $i\in a$ such that $t\notin i$ for any point t).

$\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[a]}=\mathrm{GR}\,a^a_{\mathsf{Strd}}$. $\mathcal{L}\in\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[a]}\Leftrightarrow\forall i\in n\colon\mathcal{L}_i\sqsupseteq a\Leftrightarrow\forall i\in a\colon\mathcal{L}_i\not\asymp a\Leftrightarrow\mathcal{L}\in\mathrm{GR}\,a^a_{\mathsf{Strd}}$. □

Corollary 18.56. a^a_{Strd} isn't an atom when a is a nontrivial ultrafilter.

Corollary 18.57. Staroidal product of an infinite indexed family of ultrafilters may be non-atomic.

Proposition 18.58. $\mathrm{id}^{\mathsf{Strd}}_{a[n]}$ is determined by the value of $\upuparrows\mathrm{id}^{\mathsf{Strd}}_{a[n]}$. Moreover $\mathrm{id}^{\mathsf{Strd}}_{a[n]}=\downdownarrows\upuparrows\mathrm{id}^{\mathsf{Strd}}_{a[n]}$.

Proof. Use general properties of upgrading and downgrading (proposition 17.63). □

Lemma 18.59. $\mathcal{L}\in\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ iff $\bigcup_{i\in n}\mathcal{L}_i\cup a$ has finite intersection property (for primary filtrators).

Proof. $\mathcal{L}\in\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}\Leftrightarrow\prod_{i\in n}\mathcal{L}\sqcap a\neq 0^{\mathfrak{F}}\Leftrightarrow\forall X\in\prod_{i\in n}\mathcal{L}\sqcap a\colon X\neq\emptyset$ what is equivalent of $\bigcup_{i\in n}\mathcal{L}_i\cup a$ having finite intersection property. □

Proposition 18.60. $\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ is determined by the value of $\downdownarrows\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$, moreover $\mathrm{ID}^{\mathsf{Strd}}_{a[n]}=\upuparrows\downdownarrows\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ (for primary filtrators).