Theorem 17.223. For every filters a_0, a_1, b_0, b_1 we have

$$a_0 \times^{\mathsf{FCD}} b_0 \left[f \times^{(C)} g \right] a_1 \times^{\mathsf{FCD}} b_1 \Leftrightarrow a_0 \times^{\mathsf{RLD}} b_0 \left[f \times^{(A)} g \right] a_1 \times^{\mathsf{RLD}} b_1.$$

Proof. $a_0 \times^{\mathsf{RLD}} b_0 \left[f \times^{(A)} g \right] a_1 \times^{\mathsf{RLD}} b_1 \Leftrightarrow \forall A_0 \in a_0, B_0 \in b_0, A_1 \in a_1, B_1 \in b_1 \colon A_0 \times B_0 \left[f \times^{(A)} g \right]^* A_1 \times B_1$.

$A_0 \times B_0 \left[f \times^{(A)} g \right]^* A_1 \times B_1 \Leftrightarrow A_0 \times B_0 \left[f \times^{(C)} g \right]^* A_1 \times B_1 \Leftrightarrow A_0 \left[f \right]^* A_1 \wedge B_0 \left[g \right]^* B_1$. (Here by $A_0 \times B_0 \left[f \times^{(C)} g \right]^* A_1 \times B_1$ I mean $\uparrow^{\mathsf{FCD}(\operatorname{Base} a; \operatorname{Base} b)} (A_0 \times B_0) \left[f \times^{(C)} g \right] \uparrow^{\mathsf{FCD}(\operatorname{Base} a; \operatorname{Base} b)} (A_1 \times B_1)$.)

Thus it is equivalent to $a_0 \left[f \right] a_1 \wedge b_0 \left[g \right] b_1$ that is $a_0 \times^{\mathsf{FCD}} b_0 \left[f \times^{(C)} g \right]^* a_1 \times^{\mathsf{FCD}} b_1$.

(It was used the theorem 17.151.) □

Can the above theorem be generalized for the infinitary case?

17.16 Coordinate-wise continuity

Theorem 17.224. Let μ and ν be indexed (by some index set n) families of endomorphisms for a quasi-invertible dagger category with star-morphisms, and $f_i \in \operatorname{Mor}(\operatorname{Ob} \mu_i; \operatorname{Ob} \nu_i)$ for every $i \in n$. Then:

1. $\forall i \in n \colon f_i \in \mathrm{C}(\mu_i; \nu_i) \Rightarrow \prod^{(C)} f \in \mathrm{C}\left(\prod^{(C)} \mu; \prod^{(C)} \nu \right)$;
2. $\forall i \in n \colon f_i \in \mathrm{C}'(\mu_i; \nu_i) \Rightarrow \prod^{(C)} f \in \mathrm{C}'\left(\prod^{(C)} \mu; \prod^{(C)} \nu \right)$;
3. $\forall i \in n \colon f_i \in \mathrm{C}''(\mu_i; \nu_i) \Rightarrow \prod^{(C)} f \in \mathrm{C}''\left(\prod^{(C)} \mu; \prod^{(C)} \nu \right)$.

Proof. Using the corollary 17.140:

1. $\forall i \in n \colon f_i \in \mathrm{C}(\mu_i; \nu_i) \Leftrightarrow \forall i \in n \colon f_i \circ \mu_i \sqsubseteq \nu_i \circ f_i \Rightarrow \prod^{(C)}_{i \in n} (f_i \circ \mu_i) \sqsubseteq \prod^{(C)}_{i \in n} (\nu_i \circ f_i) \Leftrightarrow \left(\prod^{(C)} f \right) \circ \left(\prod^{(C)} \mu \right) \sqsubseteq \left(\prod^{(C)} \nu \right) \circ \left(\prod^{(C)} f \right) \Leftrightarrow \prod^{(C)} f \in \mathrm{C}\left(\prod^{(C)} \mu; \prod^{(C)} \nu \right)$.
2. $\forall i \in n \colon f_i \in \mathrm{C}'(\mu_i; \nu_i) \Leftrightarrow \forall i \in n \colon \mu_i \sqsubseteq f_i^\dagger \circ \nu_i \circ f_i \Rightarrow \prod^{(C)} \mu \sqsubseteq \prod^{(C)}_{i \in n} \left(f_i^\dagger \circ \nu_i \circ f_i \right) \Leftrightarrow \prod^{(C)} \mu \sqsubseteq \left(\prod^{(C)}_{i \in n} f_i^\dagger \right) \circ \left(\prod^{(C)}_{i \in n} \nu_i \right) \circ \left(\prod^{(C)}_{i \in n} f_i \right) \Leftrightarrow \prod^{(C)} \mu \sqsubseteq \left(\prod^{(C)}_{i \in n} f_i \right)^\dagger \circ \left(\prod^{(C)}_{i \in n} \nu_i \right) \circ \left(\prod^{(C)}_{i \in n} f_i \right) \Leftrightarrow \prod^{(C)} f \in \mathrm{C}'\left(\prod^{(C)} \mu; \prod^{(C)} \nu \right)$.
3. $\forall i \in n \colon f_i \in \mathrm{C}''(\mu_i; \nu_i) \Leftrightarrow \forall i \in n \colon f_i \circ \mu_i \circ f_i^\dagger \sqsubseteq \nu_i \Rightarrow \prod^{(C)}_{i \in n} \left(f_i \circ \mu_i \circ f_i^\dagger \right) \sqsubseteq \prod^{(C)}_{i \in n} \nu_i \Leftrightarrow \prod^{(C)}_{i \in n} f_i \circ \prod^{(C)}_{i \in n} \mu_i \circ \prod^{(C)}_{i \in n} f_i^\dagger \sqsubseteq \prod^{(C)}_{i \in n} \nu_i \Leftrightarrow \prod^{(C)}_{i \in n} f_i \circ \prod^{(C)}_{i \in n} \mu_i \circ \left(\prod^{(C)}_{i \in n} f_i \right)^\dagger \sqsubseteq \prod^{(C)}_{i \in n} \nu_i \Leftrightarrow \prod^{(C)}_{i \in n} f_i \in \mathrm{C}''\left(\prod^{(C)} \mu; \prod^{(C)} \nu \right)$. □

Theorem 17.225. Let μ and ν be indexed (by some index set n) families of endofuncoids, and $f_i \in \mathsf{FCD}(\operatorname{Ob} \mu_i; \operatorname{Ob} \nu_i)$ for every $i \in n$. Then:

1. $\forall i \in n \colon f_i \in \mathrm{C}(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}\left(\prod^{(A)} \mu; \prod^{(A)} \nu \right)$;
2. $\forall i \in n \colon f_i \in \mathrm{C}'(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}'\left(\prod^{(A)} \mu; \prod^{(A)} \nu \right)$;
3. $\forall i \in n \colon f_i \in \mathrm{C}''(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}''\left(\prod^{(A)} \mu; \prod^{(A)} \nu \right)$.

Proof. Similar to the previous theorem. □

Theorem 17.226. Let μ and ν be indexed (by some index set n) families of pointfree endofuncoids between posets with least elements, and $f_i \in \mathsf{FCD}(\operatorname{Ob} \mu_i; \operatorname{Ob} \nu_i)$ for every $i \in n$. Then:

1. $\forall i \in n \colon f_i \in \mathrm{C}(\mu_i; \nu_i) \Rightarrow \prod^{(S)} f \in \mathrm{C}\left(\prod^{(S)} \mu; \prod^{(S)} \nu \right)$;