Then $\forall X\in\langle\Pr_i\rangle\mathrm{GR}\,a, Y\in\langle\Pr_i\rangle\mathrm{GR}\,b\colon X\,[f_i]^*\,Y$.

Thus $\Pr_i^{\mathsf{RLD}}a\,[f_i]\,\Pr_i^{\mathsf{RLD}}b$. So

$$\forall i\in\operatorname{dom}f\colon\Pr_i^{\mathsf{RLD}}a\,[f_i]\,\Pr_i^{\mathsf{RLD}}b$$

and thus $a\left[f\times^{(A)}g\right]b$. □

Remark 17.189. It seems that the proof of the above theorem can be simplified using cross-composition product.

Theorem 17.190. $\prod_{i\in n}^{(A)}(g_i\circ f_i)=\prod^{(A)}g\circ\prod^{(A)}f$ for indexed (by an index set n) families f and g of funcoids such that $\forall i\in n\colon\operatorname{Dst}f_i=\operatorname{Src}g_i$.

Proof. Let a, b be ultrafilters on $\prod_{i\in n}\operatorname{Src}f_i$ and $\prod_{i\in n}\operatorname{Dst}g_i$ correspondingly,

$a\left[\prod_{i\in n}^{(A)}(g_i\circ f_i)\right]b\Leftrightarrow\forall i\in\operatorname{dom}f\colon\langle\Pr_i\rangle a\,[g_i\circ f_i]\,\langle\Pr_i\rangle b\Leftrightarrow\forall i\in\operatorname{dom}f\exists C\in\operatorname{atoms}^{\mathfrak{F}(\operatorname{Dst}f_i)}\colon(\langle\Pr_i\rangle a\,[f_i]\,C\wedge C\,[g_i]\,\langle\Pr_i\rangle b)\Leftrightarrow\forall i\in\operatorname{dom}f\exists c\in\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in n:\operatorname{Dst}f)}\colon(\langle\Pr_i\rangle a\,[f_i]\,\langle\Pr_i\rangle c\wedge\langle\Pr_i\rangle c\,[g_i]\,\langle\Pr_i\rangle b)\Leftarrow\exists c\in\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in n:\operatorname{Dst}f)}\forall i\in\operatorname{dom}f\colon(\langle\Pr_i\rangle a\,[f_i]\,\langle\Pr_i\rangle c\wedge\langle\Pr_i\rangle c\,[g_i]\,\langle\Pr_i\rangle b)\Leftrightarrow\exists c\in\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in n:\operatorname{Dst}f)}\colon\left(a\left[\prod^{(A)}f\right]c\wedge c\left[\prod^{(A)}g\right]b\right)\Leftrightarrow a\left[\prod^{(A)}g\circ\prod^{(A)}f\right]b$.

Let

$$\forall i\in\operatorname{dom}f\exists c\in\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in n:\operatorname{Dst}f)}\colon(\langle\Pr_i\rangle a\,[f_i]\,\langle\Pr_i\rangle c\wedge\langle\Pr_i\rangle c\,[g_i]\,\langle\Pr_i\rangle b).$$

Then there exists $c'\in\left(\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in n:\operatorname{Dst}f)}\right)^n$ such that

$$\forall i\in\operatorname{dom}f\colon(\langle\Pr_i\rangle a\,[f_i]\,\langle\Pr_i\rangle c'_i\wedge\langle\Pr_i\rangle c'_i\,[g_i]\,\langle\Pr_i\rangle b).$$

Then take $c''=\prod^{\mathsf{RLD}}c'$. Then $\forall i\in\operatorname{dom}f\colon(\langle\Pr_i\rangle a\,[f_i]\,\langle\Pr_i\rangle c''_i\wedge\langle\Pr_i\rangle c''_i\,[g_i]\,\langle\Pr_i\rangle b)$. Thus

$$\exists c\in\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in n:\operatorname{Dst}f)}\forall i\in\operatorname{dom}f\colon(\langle\Pr_i\rangle a\,[f_i]\,\langle\Pr_i\rangle c\wedge\langle\Pr_i\rangle c\,[g_i]\,\langle\Pr_i\rangle b).$$

We have $a\left[\prod_{i\in n}^{(A)}(g_i\circ f_i)\right]b\Leftrightarrow a\left[\prod^{(A)}g\circ\prod^{(A)}f\right]b$. □

Corollary 17.191. $\left(\prod^{(A)}f_{k-1}\right)\circ\ldots\circ\left(\prod^{(A)}f_0\right)=\prod_{i\in n}^{(A)}(f_{k-1}\circ\ldots\circ f_0)$ for every n-indexed families $f_0,\ldots,f_{n-1}$ of composable funcoids.

Proposition 17.192. $\prod^{\mathsf{RLD}}a\left[\prod^{(A)}f\right]\prod^{\mathsf{RLD}}b\Leftrightarrow\forall i\in\operatorname{dom}f\colon a_i\,[f_i]\,b_i$ for an indexed family f of funcoids and indexed families a and b of filters where $a_i\in\mathfrak{F}(\operatorname{Src}f_i)$, $b_i\in\mathfrak{F}(\operatorname{Dst}f_i)$ for every $i\in\operatorname{dom}f$.

Proof. If $a_i=0$ or $b_i=0$ for some i our theorem is obvious. We will take $a_i\neq 0$ and $b_i\neq 0$, thus there exist

$$x\in\operatorname{atoms}\prod^{\mathsf{RLD}}a,\quad y\in\operatorname{atoms}\prod^{\mathsf{RLD}}b.$$

$\prod^{\mathsf{RLD}}a\left[\prod^{(A)}f\right]\prod^{\mathsf{RLD}}b\Leftrightarrow\exists x\in\operatorname{atoms}\prod^{\mathsf{RLD}}a, y\in\operatorname{atoms}\prod^{\mathsf{RLD}}b\colon x\left[\prod^{(A)}f\right]y\Leftrightarrow\exists x\in\operatorname{atoms}\prod^{\mathsf{RLD}}a, y\in\operatorname{atoms}\prod^{\mathsf{RLD}}b\forall i\in\operatorname{dom}f\colon\langle\Pr_i\rangle x\,[f_i]\,\langle\Pr_i\rangle y\Leftrightarrow\forall i\in\operatorname{dom}f\exists x\in\operatorname{atoms}a_i, y\in\operatorname{atoms}b_i\colon x\,[f_i]\,y\Leftrightarrow\forall i\in\operatorname{dom}f\colon a_i\,[f_i]\,b_i$. □

Theorem 17.193. $\left\langle\prod^{(A)}f\right\rangle x=\prod_{i\in\operatorname{dom}f}^{\mathsf{RLD}}\langle f_i\rangle\Pr_i^{\mathsf{RLD}}x$ for an indexed family f of funcoids and $x\in\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in\operatorname{dom}f:\operatorname{Src}f_i)}$ for every $n\in\operatorname{dom}f$.

Proof. For every ultrafilter $y\in\mathfrak{F}\left(\prod_{i\in\operatorname{dom}f}\operatorname{Dst}f_i\right)$ we have: