Proposition 17.183. $\prod^{\mathsf{RLD}*} a = 0^{\mathsf{RLD}(\lambda i\in n:\mathrm{Base}(a_i))}$ if a_i is the non-proper filter for some $i\in n$.

Proof. Take $A_i=\emptyset$ and $m=\{i\}$. Then $\prod_{i\in n}\left(\left\{\begin{array}{ll} A_i & \text{if } i\in m\\ \mathrm{Base}(a_i) & \text{if } i\in n\setminus m\end{array}\right.\right)=\emptyset$. □

Example 17.184. There exists an indexed family a of principal filters such that $\prod^{\mathsf{RLD}*} a$ is non-principal.

Proof. Let $n=\mathbb{N}$. Let $\mathrm{Base}(a_i)=\mathbb{R}$ and each a_i be a principal filter corresponding to a two-element set.

Every $\prod_{i\in n}\left(\left\{\begin{array}{ll} A_i & \text{if } i\in m\\ \mathrm{Base}(a_i) & \text{if } i\in n\setminus m\end{array}\right.\right)$ has at least $\mathfrak{c}^n\geqslant\mathfrak{c}$ elements.

There are elements $\prod^{\mathsf{RLD}} a$ with cardinality $2^n=n$. They can't be elements of $\prod^{\mathsf{RLD}*} a$ because $n=\omega<\mathfrak{c}$. □

Corollary 17.185. There exists an indexed family a of principal filters such that $\prod^{\mathsf{RLD}*} a\neq\prod^{\mathsf{RLD}} a$.

Proof. Because $\prod^{\mathsf{RLD}} a$ is principal. □

Proposition 17.186. $\mathrm{Pr}^{\mathsf{RLD}}_k\prod^{\mathsf{RLD}*} x=x_k$ for every indexed family x of proper filters.

Proof. $\mathrm{Pr}^{\mathsf{RLD}}_k\prod^{\mathsf{RLD}*} x=\langle\mathrm{Pr}_k\rangle\mathrm{GR}\prod^{\mathsf{RLD}*} x=x_k$. □

17.13 Subatomic product of funcoids

Definition 17.187. Let f be an indexed family of funcoids. Then $\prod^{(A)} f$ (*subatomic product*) is a funcoid $\prod_{i\in\mathrm{dom}\, f}\mathrm{Src}\, f_i\rightarrow\prod_{i\in\mathrm{dom}\, f}\mathrm{Dst}\, f_i$ such that for every $a\in\mathrm{atoms}^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Src}\, f_i)}$, $b\in\mathrm{atoms}^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Dst}\, f_i)}$

$$a\left[\prod^{(A)} f\right]b\Leftrightarrow\forall i\in\mathrm{dom}\, f:\mathrm{Pr}^{\mathsf{RLD}}_i a\,[f_i]\,\mathrm{Pr}^{\mathsf{RLD}}_i b.$$

Proposition 17.188. The funcoid $\prod^{(A)} f$ exists.

Proof. To prove that $\prod^{(A)} f$ exists we need to prove (for every $a\in\mathrm{atoms}^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Src}\, f_i)}$, $b\in\mathrm{atoms}^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Dst}\, f_i)}$)

$$\forall X\in\mathrm{GR}\, a,Y\in\mathrm{GR}\, b\exists x\in\mathrm{atoms}\uparrow^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Src}\, f_i)}X,y\in\mathrm{atoms}\uparrow^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Dst}\, f_i)}Y:x\left[\prod^{(A)} f\right]y\Rightarrow a\left[\prod^{(A)} f\right]b.$$

Let $\forall X\in\mathrm{GR}\, a$, $Y\in\mathrm{GR}\, b\exists x\in\mathrm{atoms}\uparrow^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Src}\, f_i)}X$, $y\in\mathrm{atoms}\uparrow^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Dst}\, f_i)}Y$: $x\left[\prod^{(A)} f\right]y$.

Then

$$\forall X\in\mathrm{GR}\, a,Y\in\mathrm{GR}\, b\exists x\in\mathrm{atoms}\uparrow^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Src}\, f_i)}X,y\in\mathrm{atoms}\uparrow^{\mathsf{RLD}(\lambda i\in\mathrm{dom}\, f:\mathrm{Dst}\, f_i)}Y\forall i\in\mathrm{dom}\, f:\mathrm{Pr}^{\mathsf{RLD}}_i x\,[f_i]\,\mathrm{Pr}^{\mathsf{RLD}}_i y.$$

Then because $\mathrm{Pr}^{\mathsf{RLD}}_i x\in\mathrm{atoms}\uparrow^{\mathrm{Src}\, f_i}\mathrm{Pr}_i X$ and likewise for y:

$\forall X\in\mathrm{GR}\, a,Y\in\mathrm{GR}\, b\forall i\in\mathrm{dom}\, f\exists x\in\mathrm{atoms}\uparrow^{\mathrm{Src}\, f_i}\mathrm{Pr}_i X,y\in\mathrm{atoms}\uparrow^{\mathrm{Dst}\, f_i}\mathrm{Pr}_i Y:x\,[f_i]\,y$.

Thus $\forall X\in\mathrm{GR}\, a,Y\in\mathrm{GR}\, b\forall i\in\mathrm{dom}\, f:\uparrow^{\mathrm{Src}\, f_i}\mathrm{Pr}_i X\,[f_i]\uparrow^{\mathrm{Dst}\, f_i}\mathrm{Pr}_i Y$;

$\forall X\in\mathrm{GR}\, a,Y\in\mathrm{GR}\, b\forall i\in\mathrm{dom}\, f:\mathrm{Pr}_i X\,[f_i]^*\,\mathrm{Pr}_i Y$.