17.12 Multireloids

Definition 17.164. I will call a *multireloid* of the form $A = A_{i\in n}$, where every each A_i is a set, a pair $(f; A)$ where f is a filter on the set $\prod A$.

Definition 17.165. I will denote $\operatorname{Obj}(f; A) = A$ and $\operatorname{GR}(f; A) = f$ for every multireloid $(f; A)$.

I will denote $\mathsf{RLD}(A)$ the set of multireloids of the form A.

The multireloid $\uparrow^{\mathsf{RLD}(A)} F$ for a relation F is defined by the formulas:

$$\operatorname{Obj} \uparrow^{\mathsf{RLD}(A)} F = A \quad \text{and} \quad \operatorname{GR} \uparrow^{\mathsf{RLD}(A)} F = \uparrow^{\prod A} F.$$

Let a be a multireloid of the form A and $\operatorname{dom} A = n$.

Let every f_i be a reloid with $\operatorname{Src} f_i = A_i$.

The star-composition of a with f is a multireloid of the form $\lambda i \in \operatorname{dom} A: \operatorname{Dst} f_i$ defined by the formulas:

$$\operatorname{arity} \operatorname{StarComp}(a; f) = n;$$
$$\operatorname{GR} \operatorname{StarComp}(a; f) = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(A)} \operatorname{GR} \operatorname{StarComp}(A; F) \mid A \in \operatorname{GR} a, F \in \prod_{i\in n} \operatorname{GR} f_i \right\};$$
$$\operatorname{Obj}_m \operatorname{StarComp}(a; f) = \lambda i \in n: \operatorname{Dst} f_i.$$

Theorem 17.166. Multireloids with above defined compositions form a quasi-invertible category with star-morphisms.

Proof. We need to prove:

1. $\operatorname{StarComp}(\operatorname{StarComp}(m; f); g) = \operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: g_i \circ f_i)$;
2. $\operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: \operatorname{id}_{\operatorname{Obj}_m i}) = m$;
3. $b \not\asymp \operatorname{StarComp}(a; f) \Leftrightarrow a \not\asymp \operatorname{StarComp}(b; f^\dagger)$

(the rest is obvious).

Really,

1. Using properties of generalized filter bases, $\operatorname{StarComp}(\operatorname{StarComp}(a; f); g) = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}} \operatorname{StarComp}(B; G) \mid B \in \operatorname{GR} \operatorname{StarComp}(a; f), G \in \prod_{i\in n} \operatorname{GR} g_i \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}} \operatorname{StarComp}(\operatorname{StarComp}(A; F); G) \mid A \in \operatorname{GR} a, F \in \prod_{i\in n} f_i, G \in \prod_{i\in n} g_i \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}} \operatorname{StarComp}(A; G \circ F) \mid A \in \operatorname{GR} a, F \in \prod_{i\in n} f_i, G \in \prod_{i\in n} g_i \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}} \operatorname{StarComp}(A; H) \mid A \in \operatorname{GR} a, H \in \prod_{i\in n} \lambda i \in n: g_i \circ f_i \right\} = \operatorname{StarComp}(a; \lambda i \in n: g_i \circ f_i)$.
2. $\operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: \operatorname{id}_{\operatorname{Obj}_m i}) = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(A)} \operatorname{StarComp}(A; H) \mid A \in \operatorname{GR} m, H \in \prod_{i\in \operatorname{arity} m} \operatorname{GR} \operatorname{id}_{\operatorname{Obj}_m i} \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(A)} \operatorname{StarComp}(A; \lambda i \in \operatorname{arity} m: H_i) \mid A \in \operatorname{GR} m, H \in \prod_{i\in \operatorname{arity} m} \operatorname{GR} \operatorname{id}_{\operatorname{Obj}_m i} \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(A)} \operatorname{StarComp}(A; \lambda i \in \operatorname{arity} m: \operatorname{id}_{X_i}) \mid A \in \operatorname{GR} m, X \in \prod_{i\in \operatorname{arity} m} \operatorname{Obj}_m i \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(A)} (A \cap \prod X) \mid A \in \operatorname{GR} m, X \in \prod_{i\in \operatorname{arity} m} \operatorname{Obj}_m i \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(A)} A \mid A \in \operatorname{GR} m \right\} = m$.
3. Using properties of generalized filter bases,
   $b \not\asymp \operatorname{StarComp}(a; f) \Leftrightarrow \forall A \in \operatorname{GR} a, B \in \operatorname{GR} b, F \in \prod_{i\in n} \operatorname{GR} f_i: B \not\asymp \operatorname{StarComp}(A; F) \Leftrightarrow \forall A \in \operatorname{GR} a, B \in \operatorname{GR} b, F \in \prod_{i\in n} \operatorname{GR} f_i: B \not\asymp \left\langle \prod^{(C)} F \right\rangle A \Leftrightarrow \forall A \in \operatorname{GR} a, B \in \operatorname{GR} b, F \in \prod_{i\in n} \operatorname{GR} f_i: A \not\asymp \left\langle \left(\prod^{(C)} F \right)^{-1} \right\rangle B \Leftrightarrow \forall A \in \operatorname{GR} a, B \in \operatorname{GR} b, F \in \prod_{i\in n} \operatorname{GR} f_i: A \not\asymp \operatorname{StarComp}(B; F^\dagger) \Leftrightarrow a \not\asymp \operatorname{StarComp}(b; f^\dagger)$. □

Definition 17.167. Let f be a multireloid of the form A. Then for $i \in \operatorname{dom} A$

$$\operatorname{Pr}_i^{\mathsf{RLD}} f = \bigsqcap \langle \uparrow^{A_i} \rangle \langle \operatorname{Pr}_i \rangle \operatorname{GR} f.$$