Proof. $L\in\mathrm{GR}\left\langle\prod^{(C)} f\right\rangle\prod^{\mathsf{Strd}} a\Leftrightarrow L\in\mathrm{GR}\,\mathrm{StarComp}\left(\prod^{\mathsf{Strd}} a; f\right)\Leftrightarrow\exists y\in\prod_{i\in\mathrm{dom}\,\mathfrak{A}}\mathrm{atoms}^{\mathfrak{A}_i}\forall i\in n$: $(y_i\,[f_i]\,L_i\wedge y_i\not\asymp a_i)\Leftrightarrow\forall i\in n\exists y\in\mathrm{atoms}^{\mathfrak{A}_i}$: $(y\,[f_i]\,L_i\wedge y\not\asymp a_i)\Leftrightarrow\forall i\in n$: $a_i\,[f_i]\,L_i\Leftrightarrow\forall i\in n$: $L_i\not\asymp\langle f_i\rangle a_i\Leftrightarrow L\in\mathrm{GR}\prod^{\mathsf{Strd}}_{i\in n}\langle f_i\rangle a_i$. □

Conjecture 17.155. $\mathrm{StarComp}(a\sqcup b; f)=\mathrm{StarComp}(a; f)\sqcup\mathrm{StarComp}(b; f)$ for anchored relations a, b of a form $\mathfrak{A}$, where every $\mathfrak{A}_i$ is a distributive lattice, and an indexed family f of pointfree funcoids with $\mathrm{Src}\,f_i=\mathfrak{A}_i$.

17.11.6 Simple product of pointfree funcoids

Definition 17.156. Let f be an indexed family of pointfree funcoids with every $\mathrm{Src}\,f_i$ and $\mathrm{Dst}\,f_i$ (for all $i\in\mathrm{dom}\,f$) being a poset with least element. *Simple product* of f is

$$\prod^{(S)} f=\left(\lambda x\in\prod_{i\in\mathrm{dom}\,f}\mathrm{Src}\,f_i: \lambda i\in\mathrm{dom}\,f: \langle f_i\rangle x_i; \lambda y\in\prod_{i\in\mathrm{dom}\,f}\mathrm{Dst}\,f_i: \lambda i\in\mathrm{dom}\,f: \langle f_i^{-1}\rangle y_i\right).$$

Proposition 17.157. Simple product is a pointfree funcoid

$$\prod^{(S)} f\in\mathsf{FCD}\left(\prod_{i\in\mathrm{dom}\,f}\mathrm{Src}\,f_i; \prod_{i\in\mathrm{dom}\,f}\mathrm{Dst}\,f_i\right).$$

Proof. Let $x\in\prod_{i\in\mathrm{dom}\,f}\mathrm{Src}\,f_i$ and $y\in\prod_{i\in\mathrm{dom}\,f}\mathrm{Dst}\,f_i$. Then (take into account that $\mathrm{Src}\,f_i$ and $\mathrm{Dst}\,f_i$ are posets with least elements) $y\not\asymp\left(\lambda x\in\prod_{i\in\mathrm{dom}\,f}\mathrm{Src}\,f_i: \lambda i\in\mathrm{dom}\,f: \langle f_i\rangle x_i\right)x\Leftrightarrow y\not\asymp\lambda i\in\mathrm{dom}\,f: \langle f_i\rangle x_i\Leftrightarrow\exists i\in\mathrm{dom}\,f: y_i\not\asymp\langle f_i\rangle x_i\Leftrightarrow\exists i\in\mathrm{dom}\,f: x_i\not\asymp\langle f_i^{-1}\rangle y_i\Leftrightarrow x\not\asymp\lambda i\in\mathrm{dom}\,f: \langle f_i^{-1}\rangle y_i\Leftrightarrow x\not\asymp\left(\lambda y\in\prod_{i\in\mathrm{dom}\,f}\mathrm{Dst}\,f_i: \lambda i\in\mathrm{dom}\,f: \langle f_i^{-1}\rangle y_i\right)y$. □

Obvious 17.158. $\left\langle\prod^{(S)} f\right\rangle x=\lambda i\in\mathrm{dom}\,f: \langle f_i\rangle x_i$ for $x\in\prod\mathrm{Src}\,f_i$.

Obvious 17.159. $\left(\left\langle\prod^{(S)} f\right\rangle x\right)_i=\langle f_i\rangle x_i$ for $x\in\prod\mathrm{Src}\,f_i$.

Proposition 17.160. f_i can be restored if we know $\prod^{(S)} f$ if f_i is a family of pointfree funcoids between posets with least elements.

Proof. Let's restore the value of $\langle f_i\rangle x$ where $i\in\mathrm{dom}\,f$ and $x\in\mathrm{Src}\,f_i$.

Let $x'_i=x$ and $x'_j=0$ for $j\neq i$.

Then $\langle f_i\rangle x=\langle f_i\rangle x'_i=\left(\left\langle\prod^{(S)} f\right\rangle x'\right)_i$.

We have restored the value of $\langle f_i\rangle$. Restoring the value of $\langle f_i^{-1}\rangle$ is similar. □

Remark 17.161. In the above proposition it is not required that f_i are non-zero.

Proposition 17.162. $\left(\prod^{(S)} g\right)\circ\left(\prod^{(S)} f\right)=\prod^{(S)}_{i\in n}(g_i\circ f_i)$ for n-indexed families f and g of composable pointfree funcoids between posets with least elements.

Proof. $\left\langle\prod^{(S)}_{i\in n}(g_i\circ f_i)\right\rangle x=\lambda i\in\mathrm{dom}\,f: \langle g_i\circ f_i\rangle x_i=\lambda i\in\mathrm{dom}\,f: \langle g_i\rangle\langle f_i\rangle x_i=\left\langle\prod^{(S)} g\right\rangle\lambda i\in\mathrm{dom}\,f$: $\langle f_i\rangle x_i=\left\langle\prod^{(S)} g\right\rangle\left\langle\prod^{(S)} f\right\rangle x=\left\langle\left(\prod^{(S)} g\right)\circ\left(\prod^{(S)} f\right)\right\rangle x$.

Thus $\left\langle\prod^{(S)}_{i\in n}(g_i\circ f_i)\right\rangle=\left\langle\left(\prod^{(S)} g\right)\circ\left(\prod^{(S)} f\right)\right\rangle$.

$\left\langle\left(\prod^{(S)}(g_i\circ f_i)\right)^{-1}\right\rangle=\left\langle\left(\left(\prod^{(S)} g\right)\circ\left(\prod^{(S)} f\right)\right)^{-1}\right\rangle$ is similar. □

Corollary 17.163. $\left(\prod^{(S)} f_{k-1}\right)\circ\ldots\circ\left(\prod^{(S)} f_0\right)=\prod^{(S)}_{i\in n}(f_{k-1}\circ\ldots\circ f_0)$ for every n-indexed families $f_0,\ldots,f_{n-1}$ of composable pointfree funcoids between posets with least elements.