We need to prove it is really a pointfree funcoid.

Proof. $b \not\asymp \langle \chi f \rangle a \Leftrightarrow b \not\asymp f \circ a \Leftrightarrow a \not\asymp f^{\dagger} \circ b \Leftrightarrow a \not\asymp \langle (\chi f)^{-1} \rangle b$. □

Remark 17.136. $\langle \chi f \rangle = (f \circ -)$ is the Mor-functor[17.1] $\mathrm{Mor}(f, -)$ and we can apply Yoneda lemma to it. (See any category theory book for definitions of these terms.)

Obvious 17.137. $\langle \chi (g \circ f) \rangle a = g \circ f \circ a$ for composable morphisms f and g or a quasi-invertible category.

17.11.2 General cross-composition product

Definition 17.138. Let fix a quasi-invertible category with with star-morphisms. If f is an indexed family of morphisms from its base category, then the pointfree funcoid $\prod^{(C)} f$ (*cross-composition product* of f) from $\mathrm{StarMor}(\lambda i \in \operatorname{dom} f : \mathrm{Src}\, f_i)$ to $\mathrm{StarMor}(\lambda i \in \operatorname{dom} f : \mathrm{Dst}\, f_i)$ is defined by the formulas (for all star-morphisms a and b of these forms):

$$\left\langle \prod^{(C)} f \right\rangle a = \mathrm{StarComp}(a; f) \quad \text{and} \quad \left\langle \left(\prod^{(C)} f \right)^{-1} \right\rangle b = \mathrm{StarComp}(b; f^{\dagger}).$$

It is really a pointfree funcoid by the definition of quasi-invertible category with star-morphisms.

Theorem 17.139. $\left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) = \prod^{(C)}_{i \in n} (g_i \circ f_i)$ for every n-indexed families f and g of composable morphisms of a quasi-invertible category with star-morphisms.

Proof. $\left\langle \prod^{(C)}_{i \in n} (g_i \circ f_i) \right\rangle a = \mathrm{StarComp}(a; \lambda i \in n: g_i \circ f_i) = \mathrm{StarComp}(\mathrm{StarComp}(a; f); g)$ and $\left\langle \left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) \right\rangle a = \left\langle \prod^{(C)} g \right\rangle \left\langle \prod^{(C)} f \right\rangle a = \mathrm{StarComp}(\mathrm{StarComp}(a; f); g)$.

The rest follows from symmetry. □

Corollary 17.140. $\left(\prod^{(C)} f_{k-1} \right) \circ \ldots \circ \left(\prod^{(C)} f_0 \right) = \prod^{(C)}_{i \in n} (f_{k-1} \circ \ldots \circ f_0)$ for every n-indexed families $f_0, \ldots, f_{n-1}$ of composable morphisms of a quasi-invertible category with star-morphisms.

Proof. By math induction. □

17.11.3 Star composition of binary relations

First define *star composition* for an n-ary relation a and an n-indexed family f of binary relations as an n-ary relation complying with the formulas:

$$\mathrm{Obj}_{\mathrm{StarComp}(a; f)} = \{*\}^n;$$
$$L \in \mathrm{StarComp}(a; f) \Leftrightarrow \exists y \in a \forall i \in n: y_i\, f_i\, L_i$$

where $*$ is a unique object of the group of small binary relations considered as a category.

Proposition 17.141. $b \not\asymp \mathrm{StarComp}(a; f) \Leftrightarrow \exists x \in a, y \in b \forall j \in n: x_j\, f_j\, y_j$.

Proof. $b \not\asymp \mathrm{StarComp}(a; f) \Leftrightarrow \exists y: (y \in b \wedge y \in \mathrm{StarComp}(a; f)) \Leftrightarrow \exists y: (y \in b \wedge \exists x \in a \forall j \in n: x_j\, f_j\, y_j) \Leftrightarrow \exists x \in a, y \in b \forall j \in n: x_j\, f_j\, y_j$. □

Theorem 17.142. The group of small binary relations considered as a category together with the set of of all n-ary relations (for every small n) and the above defined star-composition form a quasi-invertible category with star-morphisms.

17.1. Also called Hom-functor.