Definition 17.129. A *quasi-invertible* category with star-morphisms is a quasi-invertible precategory with star-morphisms which is a category with star-morphisms.

Each category with star-morphisms gives rise to a category (*abrupt category*, see a remark below why I call it "abrupt"), as described below. Below for simplicity I assume that the set M and the set of our indexed families of functions are disjoint. The general case (when they are not necessarily disjoint) may be easily elaborated by the reader.

- Objects are indexed (by arity m for some $m \in M$) families of objects of the category C and an (arbitrarily chosen) object None not in this set.
- There are the following disjoint sets of morphisms:
    1. indexed (by arity m for some $m \in M$) families of morphisms of C;
    2. elements of M;
    3. the identity morphism $\mathrm{id}_{\mathrm{None}}$ on None.
- Source and destination of morphisms are defined by the formulas:
    - $\mathrm{Src}\, f = \lambda i \in \mathrm{dom}\, f: \mathrm{Src}\, f_i$;
    - $\mathrm{Dst}\, f = \lambda i \in \mathrm{dom}\, f: \mathrm{Dst}\, f_i$;
    - $\mathrm{Src}\, m = \mathrm{None}$;
    - $\mathrm{Dst}\, m = \mathrm{Obj}_m$.
- Compositions of morphisms are defined by the formulas:
    - $g \circ f = \lambda i \in \mathrm{dom}\, f: g_i \circ f_i$ for our indexed families f and g of morphisms;
    - $f \circ m = \mathrm{StarComp}(m; f)$ for $m \in M$ and a composable indexed family f;
    - $m \circ \mathrm{id}_{\mathrm{None}} = m$ for $m \in M$;
    - $\mathrm{id}_{\mathrm{None}} \circ \mathrm{id}_{\mathrm{None}} = \mathrm{id}_{\mathrm{None}}$.
- Identity morphisms for an object X are:
    - $\lambda i \in X: \mathrm{id}_{X_i}$ if $X \neq \mathrm{None}$;
    - $\mathrm{id}_{\mathrm{None}}$ if $X = \mathrm{None}$.

We need to prove it is really a category.

Proof. We need to prove:

1. Composition is associative.
2. Composition with identities complies with the identity law.

Really:

1. $(h \circ g) \circ f = \lambda i \in \mathrm{dom}\, f: (h_i \circ g_i) \circ f_i = \lambda i \in \mathrm{dom}\, f: h_i \circ (g_i \circ f_i) = h \circ (g \circ f)$;
   $g \circ (f \circ m) = \mathrm{StarComp}(\mathrm{StarComp}(m; f); g) = \mathrm{StarComp}(m; \lambda i \in \mathrm{arity}\, m: g_i \circ f_i) = \mathrm{StarComp}(m; g \circ f) = (g \circ f) \circ m$;
   $f \circ (m \circ \mathrm{id}_{\mathrm{None}}) = f \circ m = (f \circ m) \circ \mathrm{id}_{\mathrm{None}}$.
2. $m \circ \mathrm{id}_{\mathrm{None}} = m$; $\mathrm{id}_{\mathrm{Dst}\, m} \circ m = \mathrm{StarComp}(m; \lambda i \in \mathrm{arity}\, m: \mathrm{id}_{\mathrm{Obj}_m i}) = m$. □

Remark 17.130. I call the above defined category *abrupt category* because (excluding identity morphisms) it allows composition with an $m \in M$ only on the left (not on the right) so that the morphism m is "abrupt" on the right.

By $[\![x_0; \ldots; x_{n-1}]\!]$ I denote an n-tuple.

Definition 17.131. Precategory with star morphisms *induced* by a dagger precategory C is:

- The base category is C.