17.10 Star categories

Definition 17.122. A *precategory with star-morphisms* consists of

1. a precategory C (*the base precategory*);
2. a set M (*star-morphisms*);
3. a function "arity" defined on M (how many objects are connected by this star-morphism);
4. a function $\operatorname{Obj}_m: \operatorname{arity} m \to \operatorname{Obj}(C)$ defined for every $m \in M$;
5. a function (*star composition*) $(m; f) \mapsto \operatorname{StarComp}(m; f)$ defined for $m \in M$ and f being an $(\operatorname{arity} m)$-indexed family of morphisms of C such that $\forall i \in \operatorname{arity} m: \operatorname{Src} f_i = \operatorname{Obj}_m i$ ($\operatorname{Src} f_i$ is the source object of the morphism f_i) such that $\operatorname{arity} \operatorname{StarComp}(m; f) = \operatorname{arity} m$

such that it holds:

1. $\operatorname{StarComp}(m; f) \in M$;
2. (*associativity law*)

$$\operatorname{StarComp}(\operatorname{StarComp}(m; f); g) = \operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: g_i \circ f_i).$$

The meaning of the set M is an extension of C having as morphisms things with arbitrary (possibly infinite) indexed set Obj_m of objects, not just two objects as morphisms of C have only source and destination.

Definition 17.123. I will call Obj_m the *form* of the star-morphism m.

(Having fixed a precategory with star-morphisms) I will denote $\operatorname{StarMor}(P)$ the set of star-morphisms of the form P.

Proposition 17.124. The sets $\operatorname{StarMor}(P)$ are disjoint (for different P).

Proof. If two star-morphisms have different forms, they are clearly not equal. □

Definition 17.125. A *category with star-morphisms* is a precategory with star-morphisms whose base is a category and the following equality (*the law of composition with identity*) holds for every star-morphism m:

$$\operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: 1_{\operatorname{Obj}_m i}) = m.$$

Definition 17.126. A *partially ordered precategory with star-morphisms* is a category with star-morphisms, whose base precategory is a partially ordered precategory and every set $\operatorname{StarMor}(X)$ is partially ordered for every X, such that:

$$m_0 \sqsubseteq m_1 \wedge f_0 \sqsubseteq f_1 \Rightarrow \operatorname{StarComp}(m_0; f_0) \sqsubseteq \operatorname{StarComp}(m_1; f_1)$$

for every $m_0, m_1 \in M$ such that $\operatorname{Obj}_{m_0} = \operatorname{Obj}_{m_1}$ and indexed families f_0 and f_1 of morphisms such that

$$\forall i \in \operatorname{arity} m: \operatorname{Src} f_0 i = \operatorname{Src} f_1 i = \operatorname{Obj}_{m_0} i = \operatorname{Obj}_{m_1} i \quad \text{and} \quad \forall i \in \operatorname{arity} m: \operatorname{Dst} f_0 i = \operatorname{Dst} f_1 i.$$

Definition 17.127. A *partially ordered category with star-morphisms* is a category with star-morphisms which is also a partially ordered precategory with star-morphisms.

Definition 17.128. A *quasi-invertible* precategory with star-morphisms is a partially ordered precategory with star-morphisms whose base precategory is a quasi-invertible precategory, such that for every index set n, star-morphisms a and b of arity n, and an n-indexed family f of morphisms of the base precategory it holds

$$b \not\asymp \operatorname{StarComp}(a; f) \Leftrightarrow a \not\asymp \operatorname{StarComp}(b; f^\dagger).$$

(Here $f^\dagger = \lambda i \in \operatorname{dom} f: (f_i)^\dagger$.)