Proof. We need to prove that $\prod^{(D)} F$ is an upper set. Let $a \in \prod^{(D)} F$ and an anchored relation $b \sqsupseteq a$ of the same form as a. We have $a = \operatorname{uncurry} z$ for some $z \in \prod F$ that is $a(i; j) = (z i) j$ for all $i \in \operatorname{dom} F$ and $j \in \operatorname{dom} F_i$ where $z i \in F_i$. Also $b(i; j) \sqsupseteq a(i; j)$. Thus $(\operatorname{curry} b) i \sqsupseteq z i$; $\operatorname{curry} b \in \prod F$ because every F_i is an upper set and so $b \in \prod^{(D)} F$. □

Proposition 17.116. Let F be an indexed family of anchored relations and every $(\operatorname{form} F)_i$ is a join-semilattice.

1. $\prod^{(D)} F$ is a prestaroid if every F_i is a prestaroid.
2. $\prod^{(D)} F$ is a staroid if every F_i is a staroid.
3. $\prod^{(D)} F$ is a completary staroid if every F_i is a completary staroid.

Proof.

1. Let $q \in \operatorname{arity} \prod^{(D)} F$ that is $q = (i; j)$ where $i \in \operatorname{dom} F$, $j \in \operatorname{arity} F_i$; let

$$L \in \prod \left(\left(\operatorname{form} \prod^{(D)} F \right) |_{(\operatorname{arity} \prod^{(D)} F) \setminus \{q\}} \right)$$

that is $L_{(i'; j')} \in \left(\operatorname{form} \prod^{(D)} F \right)_{(i'; j')}$ for every $(i'; j') \in \left(\operatorname{arity} \prod^{(D)} F \right) \setminus \{q\}$, that is $L_{(i'; j')} \in (\operatorname{form} F_i)_j$. We have $X \in \left(\operatorname{form} \prod^{(D)} F \right)_{(i; j)} \Leftrightarrow X \in (\operatorname{form} F_i)_j$. So

$$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\operatorname{form} F_i)_j \mid L \cup \{((i; j); X)\} \in \operatorname{GR} \prod^{(D)} F \right\};$$

$$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\operatorname{form} F_i)_j \mid \exists z \in \prod (\operatorname{GR} \circ F) \colon L \cup \{((i; j); X)\} = \operatorname{uncurry} z \right\};$$

$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\operatorname{form} F_i)_j \mid \exists z \in \prod \left((\operatorname{GR} \circ F)|_{(\operatorname{arity} \prod^{(D)} F) \setminus \{(i; j)\}} \right), v \in \operatorname{GR} F_i \colon (L = \operatorname{uncurry} z \wedge v_j = X) \right\}$;

$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\operatorname{form} F_i)_j \mid \exists z \in \prod \left((\operatorname{GR} \circ F)|_{(\operatorname{arity} \prod^{(D)} F) \setminus \{(i; j)\}} \right) \colon L = \operatorname{uncurry} z \wedge \exists v \in \operatorname{GR} F_i \colon v_j = X \right\}$.

If $\exists z \in \prod \left((\operatorname{GR} \circ F)|_{(\operatorname{arity} \prod^{(D)} F) \setminus \{(i; j)\}} \right) \colon L = \operatorname{uncurry} z$ is false then $\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \emptyset$ is a free star. We can assume it is true. So

$$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \{ X \in (\operatorname{form} F_i)_j \mid \exists v \in \operatorname{GR} F_i \colon v_j = X \}.$$

Thus

$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \{ X \in (\operatorname{form} F_i)_j \mid \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon K \cup \{(j; X)\} \in \operatorname{GR} F_i \} = \{ X \in (\operatorname{form} F_i)_j \mid \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon X \in (\operatorname{val} F_i)_j K \}$.

Thus $A \sqcup B \in \left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L \Leftrightarrow \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon A \sqcup B \in (\operatorname{val} F_i)_j K \Leftrightarrow \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon (A \in (\operatorname{val} F_i)_j K \vee B \in (\operatorname{val} F_i)_j K) \Leftrightarrow \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon A \in (\operatorname{val} F_i)_j K \vee \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon B \in (\operatorname{val} F_i)_j K \Leftrightarrow A \in \left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L \vee B \in \left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L$. Least element 0 is not in $\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L$ because $K \cup \{(j; 0)\} \notin \operatorname{GR} F_i$.