Thus, by the definition of completary staroid, $L_0 \in \mathrm{GR}\, f \vee L_1 \in \mathrm{GR}\, f$ that is

$$X_0 \in (\mathrm{val}\, f)_i K \vee X_1 \in (\mathrm{val}\, f)_i K.$$

So $(\mathrm{val}\, f)_i K$ is a free star (taken into account that $z_i = 0^{(\mathrm{form}\, f)_i} \Rightarrow z \notin \mathrm{GR}\, f$ and that $(\mathrm{val}\, f)_i K$ is an upper set). □

Exercise 17.2. Write a simplified proof for the case if every $(\mathrm{form}\, f)_i$ is a join-semilattice.

Lemma 17.57. Every finitary prestaroid is completary.

Proof. $\exists c \in \{0, 1\}^n: (\lambda i \in n: L_{c(i)} i) \in \mathrm{GR}\, f \Leftrightarrow \exists c \in \{0, 1\}^{n-1}: (\{(n-1; L_0(n-1))\} \cup (\lambda i \in n-1: L_{c(i)} i)) \in \mathrm{GR}\, f \vee (\{(n-1; L_1(n-1))\} \cup (\lambda i \in n-1: L_{c(i)} i)) \in \mathrm{GR}\, f \Leftrightarrow \exists c \in \{0, 1\}^{n-1}: L_0(n-1) \in (\mathrm{val}\, f)_{n-1}(\lambda i \in n-1: L_{c(i)} i) \vee L_1(n-1) \in (\mathrm{val}\, f)_{n-1}(\lambda i \in n-1: L_{c(i)} i) \Leftrightarrow \exists c \in \{0, 1\}^{n-1} \forall K \in (\mathrm{form}\, f)_{n-1}: (K \sqsupseteq L_0(n-1) \vee K \sqsupseteq L_1(n-1) \Rightarrow K \in (\mathrm{val}\, f)_{n-1}(\lambda i \in n-1: L_{c(i)} i)) \Leftrightarrow \exists c \in \{0, 1\}^{n-1} \forall K \in (\mathrm{form}\, f)_{n-1}: (K \sqsupseteq L_0(n-1) \vee K \sqsupseteq L_1(n-1) \Rightarrow \{(n-1; K)\} \cup (\lambda i \in n-1: L_{c(i)} i)) \in \mathrm{GR}\, f \Leftrightarrow \ldots \Leftrightarrow \forall K \in \prod \mathrm{form}\, f: (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \mathrm{GR}\, f)$. □

Exercise 17.3. Prove the simpler special case of the above theorem when the form is a family of join-semilattices.

Theorem 17.58. For finite arity the following are the same:

1. prestaroids;
2. staroids;
3. completary staroids.

Proof. f is a finitary prestaroid $\Rightarrow$ f is a finitary completary staroid.
f is a finitary completary staroid $\Rightarrow$ f is a finitary staroid.
f is a finitary staroid $\Rightarrow$ f is a finitary prestaroid. □

Definition 17.59. We will denote the set of staroids, prestaroids, and completary staroids of a form $\mathfrak{A}$ correspondingly as $\mathsf{Strd}(\mathfrak{A})$, $\mathsf{pStrd}(\mathfrak{A})$, and $\mathsf{cStrd}(\mathfrak{A})$.

17.4 Upgrading and downgrading a set regarding a filtrator

Let fix a filtrator $(\mathfrak{A}; \mathfrak{Z})$.

Definition 17.60. $\downdownarrows f = f \cap \mathfrak{Z}$ for every $f \in \mathscr{P}\mathfrak{A}$ (*downgrading* f).

Definition 17.61. $\upuparrows f = \{L \in \mathfrak{A} \mid \mathrm{up}\, L \subseteq f\}$ for every $f \in \mathscr{P}\mathfrak{Z}$ (*upgrading* f).

Obvious 17.62. $a \in \upuparrows f \Leftrightarrow \mathrm{up}\, a \subseteq f$ for every $f \in \mathscr{P}\mathfrak{Z}$ and $a \in \mathfrak{A}$.

Proposition 17.63. $\downdownarrows \upuparrows f = f$ if f is an upper set for every $f \in \mathscr{P}\mathfrak{Z}$.

Proof. $\downdownarrows \upuparrows f = \upuparrows f \cap \mathfrak{Z} = \{L \in \mathfrak{Z} \mid \mathrm{up}\, L \subseteq f\} = \{L \in \mathfrak{Z} \mid L \in f\} = f \cap \mathfrak{Z} = f$. □

17.4.1 Upgrading and downgrading staroids

Let fix a family $(\mathfrak{A}; \mathfrak{Z})$ of filtrators.

For a graph f of a staroid define $\downdownarrows f$ and $\upuparrows f$ taking the filtrator of $(\prod \mathfrak{A}; \prod \mathfrak{Z})$.

For a staroid f define: [TODO: Define for all anchored relations.]

$$\mathrm{form} \downdownarrows f = \mathfrak{Z} \quad \text{and} \quad \mathrm{GR} \downdownarrows f = \downdownarrows \mathrm{GR}\, f;$$
$$\mathrm{form} \upuparrows f = \mathfrak{A} \quad \text{and} \quad \mathrm{GR} \upuparrows f = \upuparrows \mathrm{GR}\, f.$$

Proposition 17.64. $(\mathrm{val} \downdownarrows f)_i L = (\mathrm{val}\, f)_i L \cap \mathfrak{Z}_i$ for every $L \in \prod \mathfrak{Z}|_{(\mathrm{arity}\, f) \setminus \{i\}}$.