Definition 17.49. A *prestaroid* is an anchored relation f between posets such that $(\operatorname{val} f)_i L$ is a free star for every $i \in \operatorname{arity} f$, $L \in \prod (\operatorname{form} f)|_{(\operatorname{arity} f) \setminus \{i\}}$.

Definition 17.50. A *staroid* is a prestaroid whose graph is an upper set (on the poset $\prod \operatorname{form}(f)$).

Proposition 17.51. If $L \in \prod \operatorname{form} f$ and $L_i = 0^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin f$ if f is a prestaroid.

Proof. Let $K = L|_{(\operatorname{arity} f) \setminus \{i\}}$. We have $0 \notin (\operatorname{val} f)_i K$; $K \cup \{(i; 0)\} \notin f$; $L \notin f$. □

Definition 17.52. *Infinitary anchored relation* is such an anchored relation whose arity is infinite; *finitary anchored relation* is such an anchored relation whose arity is finite.

Next we will define *completary staroids*. First goes the general case, next simpler case for the special case of join-semilattices instead of arbitrary posets.

Definition 17.53. A *completary staroid* is an anchored relation between posets conforming to the formulas:

1. $\forall K \in \prod \operatorname{form} f\colon (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \operatorname{GR} f) \Leftrightarrow \exists c \in \{0,1\}^n\colon (\lambda i \in n\colon L_{c(i)} i) \in \operatorname{GR} f$ for every $L_0, L_1 \in \prod \operatorname{form} f$.
2. If $L \in \prod \operatorname{form} f$ and $L_i = 0^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin \operatorname{GR} f$.

Lemma 17.54. Every graph of completary staroid is an upper set.

Proof. Let f be a completary staroid. Let $L_0 \sqsubseteq L_1$ for some $L_0, L_1 \in \prod \operatorname{form} f$ and $L_0 \in \operatorname{GR} f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n\colon L_{c(i)} i = \lambda i \in n\colon L_0 i = L_0 \in \operatorname{GR} f$ and thus $L_1 \in \operatorname{GR} f$ because $L_1 \sqsupseteq L_0 \wedge L_1 \sqsupseteq L_1$. □

Proposition 17.55. A relation between posets whose form is a family of join-semilattices is a completary staroid iff both:

1. $L_0 \sqcup L_1 \in \operatorname{GR} f \Leftrightarrow \exists c \in \{0,1\}^n\colon (\lambda i \in n\colon L_{c(i)} i) \in \operatorname{GR} f$ for every $L_0, L_1 \in \prod \operatorname{form} f$.
2. If $L \in \prod \operatorname{form} f$ and $L_i = 0^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin \operatorname{GR} f$.

Proof. Let the formulas (1) and (2) hold. Then f is an upper set: Let $L_0 \sqsubseteq L_1$ for some L_0, $L_1 \in \prod \operatorname{form} f$ and $L_0 \in f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n\colon L_{c(i)} i = \lambda i \in n\colon L_0 i = L_0 \in \operatorname{GR} f$ and thus $L_1 = L_0 \sqcup L_1 \in f$.

Thus to finish the proof it is enough to show that

$$L_0 \sqcup L_1 \in \operatorname{GR} f \Leftrightarrow \forall K \in \prod \operatorname{form} f\colon (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \operatorname{GR} f)$$

under condition that $\operatorname{GR} f$ is an upper set. But this is obvious. □

Proposition 17.56. Every completary staroid is a staroid.

Proof. Let f be a completary staroid.

Let $i \in \operatorname{arity} f$, $K \in \prod_{i \in (\operatorname{arity} f) \setminus \{i\}} (\operatorname{form} f)_i$. Let $L_0 = K \cup \{(i; X_0)\}$, $L_1 = K \cup \{(i; X_1)\}$ for some $X_0, X_1 \in \mathfrak{A}_i$.

Let

$$\forall Z \in \mathfrak{A}_i\colon (Z \sqsupseteq X_0 \wedge Z \sqsupseteq X_1 \Rightarrow Z \in (\operatorname{val} f)_i K);$$

then

$$\forall Z \in \mathfrak{A}_i\colon (Z \sqsupseteq X_0 \wedge Z \sqsupseteq X_1 \Rightarrow K \cup \{(i; Z)\} \in \operatorname{GR} f).$$

If $z \sqsupseteq L_0 \wedge z \sqsupseteq L_1$ then $z \sqsupseteq K \cup \{(i; z_i)\}$, thus taking into account that $\operatorname{GR} f$ is an upper set,

$$\forall z \in \prod \mathfrak{A}\colon (z \sqsupseteq L_0 \wedge z \sqsupseteq L_1 \Rightarrow K \cup \{(i; z_i)\} \in \operatorname{GR} f).$$
$$\forall z \in \prod \mathfrak{A}\colon (z \sqsupseteq L_0 \wedge z \sqsupseteq L_1 \Rightarrow z \in \operatorname{GR} f).$$