Proposition 17.22. If each $\mathfrak{A}_i$ is a separable poset with least element (for some index set n) then $\prod \mathfrak{A}$ is a separable poset.

Proof. Let $a \neq b$. Then $\exists i \in \operatorname{dom} \mathfrak{A}: a_i \neq b_i$. So $\exists x \in \mathfrak{A}_i: (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).
Take $y = (((\operatorname{dom} \mathfrak{A}) \setminus \{i\}) \times \{0\}) \cup \{(i; x)\}$. Then $y \not\asymp a$ and $y \asymp b$. □

Obvious 17.23. If every $\mathfrak{A}_i$ is a poset with least element 0_i, then the set of atoms of $\prod \mathfrak{A}$ is

$$\{(\{k\} \times \operatorname{atoms}^{\mathfrak{A}_k}) \cup (\lambda i \in (\operatorname{dom} \mathfrak{A}) \setminus \{k\}: 0_i) \mid k \in \operatorname{dom} \mathfrak{A}\}.$$

Proposition 17.24. If every $\mathfrak{A}_i$ is an atomistic poset with least element 0_i, then $\prod \mathfrak{A}$ is an atomistic poset.

Proof. $x_i = \bigsqcup \operatorname{atoms} x_i$ for every $x_i \in \mathfrak{A}_i$. Thus

$$x = \lambda i \in \operatorname{dom} x: x_i = \lambda i \in \operatorname{dom} x: \bigsqcup \operatorname{atoms} x_i = \bigsqcup_{i \in \operatorname{dom} x} \lambda j \in \operatorname{dom} x: \begin{cases} x_i & \text{if } j = i \\ 0_i & \text{if } j \neq i. \end{cases}$$

Take join two times. □

Corollary 17.25. If $\mathfrak{A}_i$ are atomistic posets with least elements, then $\prod \mathfrak{A}$ is atomically separable.

Proof. Proposition 3.19. □

Proposition 17.26. Let $(\mathfrak{A}_{i \in n}; \mathfrak{Z}_{i \in n})$ be a family of filtrators. Then $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a filtrator.

Proof. We need to prove that $\prod \mathfrak{Z}$ is a sub-poset of $\prod \mathfrak{A}$. First $\prod \mathfrak{Z} \subseteq \prod \mathfrak{A}$ because $\mathfrak{Z}_i \subseteq \mathfrak{A}_i$ for each $i \in n$.
Let $A, B \in \prod \mathfrak{Z}$ and $A \sqsubseteq^{\prod \mathfrak{Z}} B$. Then $\forall i \in n: A_i \sqsubseteq^{\mathfrak{Z}_i} B_i$; consequently $\forall i \in n: A_i \sqsubseteq^{\mathfrak{A}_i} B_i$ that is $A \sqsubseteq^{\prod \mathfrak{A}} B$. □

Proposition 17.27. Let $(\mathfrak{A}_{i \in n}; \mathfrak{Z}_{i \in n})$ be a family of filtrators.

1. The filtrator $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is (finitely) join-closed if every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is (finitely) join-closed.
2. The filtrator $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is (finitely) meet-closed if every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is (finitely) meet-closed.

Proof. Let every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ be finitely join-closed. Let $A, B \in \prod \mathfrak{Z}$ and $A \sqcup^{\prod \mathfrak{Z}} B$ exist. Then (by corollary 17.19) $A \sqcup^{\prod \mathfrak{Z}} B = \lambda i \in n: A_i \sqcup^{\mathfrak{Z}_i} B_i = \lambda i \in n: A_i \sqcup^{\mathfrak{A}_i} B_i = A \sqcup^{\prod \mathfrak{A}} B$.
Let now every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ be join-closed. Let $S \in \mathscr{P} \prod \mathfrak{Z}$ and $\bigsqcup^{\prod \mathfrak{Z}} S$ exist. Then (by corollary 17.19) $\bigsqcup^{\prod \mathfrak{Z}} S = \lambda i \in \operatorname{dom} \mathfrak{A}: \bigsqcup^{\mathfrak{Z}_i} \{x_i \mid x \in S\} = \lambda i \in \operatorname{dom} \mathfrak{A}: \bigsqcup^{\mathfrak{A}_i} \{x_i \mid x \in S\} = \bigsqcup^{\prod \mathfrak{A}} S$.
The rest follows from symmetry. □

Proposition 17.28. If each $(\mathfrak{A}_i; \mathfrak{Z}_i)$ where $i \in n$ (for some index set n) is a down-aligned filtrator with separable core then $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is with separable core.

Proof. Let $a \neq b$. Then $\exists i \in n: a_i \neq b_i$. So $\exists x \in \mathfrak{Z}_i: (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).
Take $y = ((n \setminus \{i\}) \times \{0\}) \cup \{(i; x)\}$. Then we have $y \not\asymp a$ and $y \asymp b$ and $y \in \mathfrak{Z}$. □

Proposition 17.29. Let every $\mathfrak{A}_i$ be a bounded lattice. Every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is a central filtrator iff $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a central filtrator.

Proof. $x \in Z(\prod \mathfrak{A}) \Leftrightarrow \exists y \in \prod \mathfrak{A}: \left(x \sqcap y = 0^{\prod \mathfrak{A}} \wedge x \sqcup y = 1^{\prod \mathfrak{A}}\right) \Leftrightarrow \exists y \in \prod \mathfrak{A} \forall i \in \operatorname{dom} \mathfrak{A}: (x_i \sqcap y_i = 0^{\mathfrak{A}_i} \wedge x_i \sqcup y_i = 1^{\mathfrak{A}_i}) \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A} \exists y \in \mathfrak{A}_i: (x_i \sqcap y = 0^{\mathfrak{A}_i} \wedge x_i \sqcup y = 1^{\mathfrak{A}_i}) \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: x_i \in Z(\mathfrak{A}_i)$.
[TODO: Finish the proof.] □

Proposition 17.30. For every element a of a product filtrator $(\prod \mathfrak{A}; \prod \mathfrak{Z})$:

1. $\operatorname{up} a = \prod_{i \in \operatorname{dom} a} \operatorname{up} a_i$;
2. $\operatorname{down} a = \prod_{i \in \operatorname{dom} a} \operatorname{down} a_i$.