Proof. If $\operatorname{dom}\mathfrak{A}=\emptyset$, then $a=b=0$, $a\asymp b$ and thus the theorem statement holds. Assume $\operatorname{dom}\mathfrak{A}\neq\emptyset$.

$a\not\asymp b\Leftrightarrow\exists c\in\prod\mathfrak{A}\setminus\{0^{\prod\mathfrak{A}}\}:(c\sqsubseteq a\wedge c\sqsubseteq b)\Leftrightarrow\exists c\in\prod\mathfrak{A}\setminus\{0^{\prod\mathfrak{A}}\}\forall i\in\operatorname{dom}\mathfrak{A}:(c_i\sqsubseteq a_i\wedge c_i\sqsubseteq b_i)\Leftrightarrow$(for the reverse implication take $c_j=0^{\mathfrak{A}_j}$ for $i\neq j$)$\Leftrightarrow\exists i\in\operatorname{dom}\mathfrak{A}, c\in\mathfrak{A}_i\setminus\{0^{\mathfrak{A}_i}\}:(c\sqsubseteq a_i\wedge c\sqsubseteq b_i)\Leftrightarrow\exists i\in\operatorname{dom}\mathfrak{A}: a_i\not\asymp b_i$. □

Proposition 17.14.

1. If $\mathfrak{A}_i$ are join-semilattices then $\mathfrak{A}$ is a join-semilattice and
$$A\sqcup B=\lambda i\in\operatorname{dom}\mathfrak{A}: Ai\sqcup Bi. \tag{17.2}$$
2. If $\mathfrak{A}_i$ are meet-semilattices then $\mathfrak{A}$ is a meet-semilattice and
$$A\sqcap B=\lambda i\in\operatorname{dom}\mathfrak{A}: Ai\sqcap Bi.$$

Proof. It is enough to prove the formula (17.2).

It's obvious that $\lambda i\in\operatorname{dom}\mathfrak{A}: Ai\sqcup Bi\sqsupseteq A, B$.

Let $C\sqsupseteq A, B$. Then (for every $i\in\operatorname{dom}\mathfrak{A}$) $Ci\sqsupseteq Ai$ and $Ci\sqsupseteq Bi$. Thus $Ci\sqsupseteq Ai\sqcup Bi$ that is $C\sqsupseteq\lambda i\in\operatorname{dom}\mathfrak{A}: Ai\sqcup Bi$. □

Corollary 17.15. If $\mathfrak{A}_i$ are lattices then $\mathfrak{A}$ is a lattice.

Obvious 17.16. If $\mathfrak{A}_i$ are distributive lattices then $\mathfrak{A}$ is a distributive lattice.

Proposition 17.17. If $\mathfrak{A}_i$ are boolean lattices then $\prod\mathfrak{A}$ is a boolean lattice.

Proof. We need to prove only that every element $a\in\prod\mathfrak{A}$ has a complement. But this complement is evidently $\lambda i\in\operatorname{dom}a:\overline{a_i}$. □

Proposition 17.18. If every $\mathfrak{A}_i$ is a poset then for every $S\in\mathscr{P}\prod\mathfrak{A}$

1. $\bigsqcup S=\lambda i\in\operatorname{dom}\mathfrak{A}:\bigsqcup\{x_i\mid x\in S\}$ whenever every $\bigsqcup\{x_i\mid x\in S\}$ exists;
2. $\bigsqcap S=\lambda i\in\operatorname{dom}\mathfrak{A}:\bigsqcap\{x_i\mid x\in S\}$ whenever every $\bigsqcap\{x_i\mid x\in S\}$ exists.

Proof. It's enough to prove the first formula.

$(\lambda i\in\operatorname{dom}\mathfrak{A}:\bigsqcup\{x_i\mid x\in S\})_i=\bigsqcup\{x_i\mid x\in S\}\sqsupseteq x_i$ for every $x\in S$ and $i\in\operatorname{dom}\mathfrak{A}$.

Let $y\sqsupseteq x$ for every $x\in S$. Then $y_i\sqsupseteq x_i$ for every $i\in\operatorname{dom}\mathfrak{A}$ and thus $y_i\sqsupseteq\bigsqcup\{x_i\mid x\in S\}=(\lambda i\in\operatorname{dom}\mathfrak{A}:\bigsqcup\{x_i\mid x\in S\})_i$ that is $y\sqsupseteq\lambda i\in\operatorname{dom}\mathfrak{A}:\bigsqcup\{x_i\mid x\in S\}$.

Thus $\bigsqcup S=\lambda i\in\operatorname{dom}\mathfrak{A}:\bigsqcup\{x_i\mid x\in S\}$ by the definition of join. □

Corollary 17.19. If $\mathfrak{A}_i$ are posets then for every $S\in\mathscr{P}\prod\mathfrak{A}$

1. $\bigsqcup S=\lambda i\in\operatorname{dom}\mathfrak{A}:\bigsqcup\{x_i\mid x\in S\}$ whenever $\bigsqcup S$ exists;
2. $\bigsqcap S=\lambda i\in\operatorname{dom}\mathfrak{A}:\bigsqcap\{x_i\mid x\in S\}$ whenever $\bigsqcap S$ exists.

Proof. It is enough to prove that (for every i) $\bigsqcup\{x_i\mid x\in S\}$ exists whenever $\bigsqcup S$ exists.

Fix $i\in\operatorname{dom}\mathfrak{A}$.

Take $y_i=(\bigsqcup S)_i$ and let prove that y_i is the least upper bound of $\{x_i\mid x\in S\}$.

y_i is an upper bound of $\{x_i\mid x\in S\}$ because $\bigsqcup S\sqsupseteq x$ and thus $(\bigsqcup S)_i\sqsupseteq x_i$ for every $x\in S$.

Let $x\in S$ and for some $t\in\mathfrak{A}_i$
$$T(t)=\lambda j\in\operatorname{dom}\mathfrak{A}:\begin{cases}t & \text{if } i=j\\ x_i & \text{if } i\neq j.\end{cases}$$
Let $t\sqsupseteq x_i$. Then $T(t)\sqsupseteq x$ for every $x\in S$. So $T(t)\sqsupseteq\bigsqcup S$ and consequently $t=T(t)_i\sqsupseteq y_i$.

So y_i is the least upper bound of $\{x_i\mid x\in S\}$. □

Corollary 17.20. If $\mathfrak{A}_i$ are complete lattices then $\mathfrak{A}$ is a complete lattice.

Obvious 17.21. If $\mathfrak{A}_i$ are complete (co-)brouwerian lattices then $\mathfrak{A}$ is a (co-)brouwerian lattice.