Proposition 17.5. $a\left[f\times^{(C)}g\right]b\Leftrightarrow a\circ f^{\dagger}\not\asymp g^{\dagger}\circ b$.

Proof. From the definition. □

Proposition 17.6. $a\left[f\times^{(C)}g\right]b\Leftrightarrow f\left[a\times^{(C)}b\right]g$.

Proof. $f\left[a\times^{(C)}b\right]g\Leftrightarrow f\circ a^{\dagger}\not\asymp b^{\dagger}\circ g\Leftrightarrow a\circ f^{\dagger}\not\asymp g^{\dagger}\circ b\Leftrightarrow a\left[f\times^{(C)}g\right]b$. □

Theorem 17.7. $\left(f\times^{(C)}g\right)^{-1}=f^{\dagger}\times^{(C)}g^{\dagger}$.

Proof. For every funcoids $a\in\operatorname{Mor}(\operatorname{Src}f;\operatorname{Src}g)$ and $b\in\operatorname{Mor}(\operatorname{Dst}f;\operatorname{Dst}g)$ we have:
$\left\langle\left(f\times^{(C)}g\right)^{-1}\right\rangle b=g^{\dagger}\circ b\circ f=\left\langle f^{\dagger}\times^{(C)}g^{\dagger}\right\rangle b$.
$\left\langle\left(\left(f\times^{(C)}g\right)^{-1}\right)^{-1}\right\rangle a=\left\langle f\times^{(C)}g\right\rangle a=g\circ a\circ f^{\dagger}=\left\langle\left(f^{\dagger}\times^{(C)}g^{\dagger}\right)^{-1}\right\rangle a$. □

Theorem 17.8. Let f, g be pointfree funcoids between filters on boolean lattices. Then for every filters $\mathcal{A}_0\in\mathfrak{F}(\operatorname{Src}f)$, $\mathcal{B}_0\in\mathfrak{F}(\operatorname{Src}g)$

$$\left\langle f\times^{(C)}g\right\rangle(\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0)=\langle f\rangle\mathcal{A}_0\times^{\mathsf{FCD}}\langle g\rangle\mathcal{B}_0.$$

Proof. [TODO: No reason to restrict to atomic filters?] For every atom $a_1\times^{\mathsf{FCD}}b_1$ ($a_1\in\operatorname{atoms}^{\operatorname{Dst}f}$, $b_1\in\operatorname{atoms}^{\operatorname{Dst}g}$) (see theorem 15.76) of the lattice of funcoids we have:
$a_1\times^{\mathsf{FCD}}b_1\not\asymp\left\langle f\times^{(C)}g\right\rangle(\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0)\Leftrightarrow\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0\left[f\times^{(C)}g\right]a_1\times^{\mathsf{FCD}}b_1\Leftrightarrow(\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0)\circ f^{-1}\not\asymp g^{-1}\circ(a_1\times^{\mathsf{FCD}}b_1)\Leftrightarrow\langle f\rangle\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0\not\asymp a_1\times^{\mathsf{FCD}}\langle g^{-1}\rangle b_1\Leftrightarrow\langle f\rangle\mathcal{A}_0\not\asymp a_1\wedge\langle g^{-1}\rangle b_1\not\asymp\mathcal{B}_0\Leftrightarrow\langle f\rangle\mathcal{A}_0\not\asymp a_1\wedge\langle g\rangle\mathcal{B}_0\not\asymp b_1\Leftrightarrow\langle f\rangle\mathcal{A}_0\times^{\mathsf{FCD}}\langle g\rangle\mathcal{B}_0\not\asymp a_1\times^{\mathsf{FCD}}b_1$. Thus $\left\langle f\times^{(C)}g\right\rangle(\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0)=\langle f\rangle\mathcal{A}_0\times^{\mathsf{FCD}}\langle g\rangle\mathcal{B}_0$ because the lattice $\mathsf{FCD}(\mathfrak{F}(\operatorname{Dst}f);\mathfrak{F}(\operatorname{Dst}g))$ is atomically separable (corollary 15.79). □

Corollary 17.9. $\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0\left[f\times^{(C)}g\right]\mathcal{A}_1\times^{\mathsf{FCD}}\mathcal{B}_1\Leftrightarrow\mathcal{A}_0\,[f]\,\mathcal{A}_1\wedge\mathcal{B}_0\,[g]\,\mathcal{B}_1$ for every $\mathcal{A}_0\in\mathfrak{F}(\operatorname{Src}f)$, $\mathcal{A}_1\in\mathfrak{F}(\operatorname{Dst}f)$, $\mathcal{B}_0\in\mathfrak{F}(\operatorname{Src}g)$, $\mathcal{B}_1\in\mathfrak{F}(\operatorname{Dst}g)$ where $\operatorname{Src}f$, $\operatorname{Dst}f$, $\operatorname{Src}g$, $\operatorname{Dst}g$ are boolean lattices.

Proof. $\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0\left[f\times^{(C)}g\right]\mathcal{A}_1\times^{\mathsf{FCD}}\mathcal{B}_1\Leftrightarrow\mathcal{A}_1\times^{\mathsf{FCD}}\mathcal{B}_1\not\asymp\left\langle f\times^{(C)}g\right\rangle(\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0)\Leftrightarrow\mathcal{A}_1\times^{\mathsf{FCD}}\mathcal{B}_1\not\asymp\langle f\rangle\mathcal{A}_0\times^{\mathsf{FCD}}\langle g\rangle\mathcal{B}_0\Leftrightarrow\mathcal{A}_1\not\asymp\langle f\rangle\mathcal{A}_0\wedge\mathcal{B}_1\not\asymp\langle g\rangle\mathcal{B}_0\Leftrightarrow\mathcal{A}_0\,[f]\,\mathcal{A}_1\wedge\mathcal{B}_0\,[g]\,\mathcal{B}_1$. □

17.2 Function spaces of posets

Definition 17.10. Let $\mathfrak{A}_i$ be a family of posets indexed by some set $\operatorname{dom}\mathfrak{A}$. We will define order of families of posets by the formula

$$a\sqsubseteq b\Leftrightarrow\forall i\in\operatorname{dom}\mathfrak{A}:a_i\sqsubseteq b_i.$$

I will call this new poset $\mathfrak{A}=\prod\mathfrak{A}$ *the function space* of posets and the above order *product order*.

Proposition 17.11. The function space for posets is also a poset.

Proof.

Reflexivity. Obvious.

Antisymmetry. Obvious.

Transitivity. Obvious. □

Obvious 17.12. $\mathfrak{A}$ has least element iff each $\mathfrak{A}_i$ has a least element. In this case

$$\min\mathfrak{A}=\prod_{i\in\operatorname{dom}\mathfrak{A}}\min\mathfrak{A}_i.$$

Proposition 17.13. $a\not\asymp b\Leftrightarrow\exists i\in\operatorname{dom}\mathfrak{A}:a_i\not\asymp b_i$ for every $a,b\in\prod\mathfrak{A}$ if every $\mathfrak{A}_i$ has least element.