Chapter 16

Convergence of funcoids

16.1 Convergence

The following generalizes the well-known notion of a filter convergent to a point or to a set:

Definition 16.1. A filter $\mathcal{F} \in \mathfrak{F}(\mathrm{Dst}\,\mu)$ *converges* to a filter $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\,\mu)$ regarding a funcoid μ ($\mathcal{F} \xrightarrow{\mu} \mathcal{A}$) iff $\mathcal{F} \sqsubseteq \langle \mu \rangle \mathcal{A}$.

Definition 16.2. A funcoid f *converges* to a filter $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\,\mu)$ regarding a funcoid μ where $\mathrm{Dst}\,f = \mathrm{Dst}\,\mu$ (denoted $f \xrightarrow{\mu} \mathcal{A}$) iff $\mathrm{im}\,f \sqsubseteq \langle \mu \rangle \mathcal{A}$ that is iff $\mathrm{im}\,f \xrightarrow{\mu} \mathcal{A}$.

Definition 16.3. A funcoid f *converges* to a filter $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\,\mu)$ on a filter $\mathcal{B} \in \mathfrak{F}(\mathrm{Src}\,f)$ regarding a funcoid μ where $\mathrm{Dst}\,f = \mathrm{Dst}\,\mu$ iff $f|_{\mathcal{B}} \xrightarrow{\mu} \mathcal{A}$.

Remark 16.4. We can define also convergence for a reloid f: $f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \mathrm{im}\,f \sqsubseteq \langle \mu \rangle \mathcal{A}$ or what is the same $f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow (\mathsf{FCD})f \xrightarrow{\mu} \mathcal{A}$.

Theorem 16.5. Let f, g be funcoids, μ, ν be endofuncoids, $\mathrm{Dst}\,f = \mathrm{Src}\,g = \mathrm{Ob}\,\mu$, $\mathrm{Dst}\,g = \mathrm{Ob}\,\nu$, $\mathcal{A} \in \mathfrak{F}(\mathrm{Ob}\,\mu)$. If $f \xrightarrow{\mu} \mathcal{A}$,

$$g|_{\langle \mu \rangle \mathcal{A}} \in \mathrm{C}(\mu \sqcap (\langle \mu \rangle \mathcal{A} \times^{\mathsf{FCD}} \langle \mu \rangle \mathcal{A}); \nu),$$

and $\langle \mu \rangle \mathcal{A} \sqsupseteq \mathcal{A}$, then $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$.

Proof. $\mathrm{im}\,f \sqsubseteq \langle \mu \rangle \mathcal{A}$; $\langle g \rangle \mathrm{im}\,f \sqsubseteq \langle g \rangle \langle \mu \rangle \mathcal{A}$; $\mathrm{im}(g \circ f) \sqsubseteq \langle g|_{\langle \mu \rangle \mathcal{A}} \rangle \langle \mu \rangle \mathcal{A}$; $\mathrm{im}(g \circ f) \sqsubseteq \langle g|_{\langle \mu \rangle \mathcal{A}} \rangle \langle \mu \sqcap (\langle \mu \rangle \mathcal{A} \times^{\mathsf{FCD}} \langle \mu \rangle \mathcal{A}) \rangle \mathcal{A}$; $\mathrm{im}(g \circ f) \sqsubseteq \langle g|_{\langle \mu \rangle \mathcal{A}} \circ (\mu \sqcap (\langle \mu \rangle \mathcal{A} \times^{\mathsf{FCD}} \langle \mu \rangle \mathcal{A})) \rangle \mathcal{A}$; $\mathrm{im}(g \circ f) \sqsubseteq \langle \nu \circ g|_{\langle \mu \rangle \mathcal{A}} \rangle \mathcal{A}$; $\mathrm{im}(g \circ f) \sqsubseteq \langle \nu \circ g \rangle \mathcal{A}$; $\mathrm{im}(g \circ f) \sqsubseteq \langle \nu \rangle \langle g \rangle \mathcal{A}$; $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$. □

Corollary 16.6. Let f, g be funcoids, μ, ν be endofuncoids, $\mathrm{Dst}\,f = \mathrm{Src}\,g = \mathrm{Ob}\,\mu$, $\mathrm{Dst}\,g = \mathrm{Ob}\,\nu$, $\mathcal{A} \in \mathfrak{F}(\mathrm{Ob}\,\mu)$. If $f \xrightarrow{\mu} \mathcal{A}$, $g \in \mathrm{C}(\mu; \nu)$, and $\langle \mu \rangle \mathcal{A} \sqsupseteq \mathcal{A}$ then $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$.

Proof. From the last theorem and theorem 10.7. □

16.2 Relationships between convergence and continuity

Lemma 16.7. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)$, $\mathcal{A} \in \mathfrak{F}(\mathrm{Ob}\,\mu)$, $\mathrm{Src}\,f = \mathrm{Ob}\,\mu$, $\mathrm{Dst}\,f = \mathrm{Ob}\,\nu$. If $f \in \mathrm{C}(\mu|_{\mathcal{A}}; \nu)$ then

$$f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}} \rangle \mathcal{A} \sqsubseteq \langle \nu \circ f \rangle \mathcal{A}.$$

Proof. $f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A} \Leftrightarrow \mathrm{im}\,f|_{\langle \mu \rangle \mathcal{A}} \sqsubseteq \langle \nu \rangle \langle f \rangle \mathcal{A} \Leftrightarrow \langle f \rangle \langle \mu \rangle \mathcal{A} \sqsubseteq \langle \nu \rangle \langle f \rangle \mathcal{A} \Leftrightarrow \langle f \circ \mu \rangle \mathcal{A} \sqsubseteq \langle \nu \circ f \rangle \mathcal{A} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}} \rangle \mathcal{A} \sqsubseteq \langle \nu \circ f \rangle \mathcal{A}$. □

Theorem 16.8. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu)$, $\mathcal{A} \in \mathfrak{F}(\mathrm{Ob}\,\mu)$, $\mathrm{Src}\,f = \mathrm{Ob}\,\mu$, $\mathrm{Dst}\,f = \mathrm{Ob}\,\nu$. If $f \in \mathrm{C}(\mu|_{\mathcal{A}}; \nu)$ then $f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A}$.