¬(2)⇒¬(1). Suppose $\langle f\rangle a\notin\operatorname{atoms}^{\mathfrak{B}}\cup\{0^{\mathfrak{B}}\}$ for some $a\in\operatorname{atoms}^{\mathfrak{A}}$. Then there exist two atoms $p\neq q$ such that $\langle f\rangle a\sqsupseteq p\wedge\langle f\rangle a\sqsupseteq q$. Consequently $p\sqcap\langle f\rangle a\neq 0^{\mathfrak{B}}$; $a\sqcap\langle f^{-1}\rangle p\neq 0^{\mathfrak{A}}$; $a\sqsubseteq\langle f^{-1}\rangle p$; $\langle f\circ f^{-1}\rangle p=\langle f\rangle\langle f^{-1}\rangle p\sqsupseteq\langle f\rangle a\sqsupseteq q$ (by proposition 15.14 because $\mathfrak{B}$ is separable by proposition 3.22); $\langle f\circ f^{-1}\rangle p\not\sqsubseteq p$ and $\langle f\circ f^{-1}\rangle p\neq 0^{\mathfrak{B}}$. So it cannot be $f\circ f^{-1}\sqsubseteq\operatorname{id}^{\mathsf{FCD}(\mathfrak{B})}$. □

Theorem 15.105. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ be primary filtrators over a boolean lattice. A pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is monovalued iff

$$\forall I,J\in\mathfrak{Z}_1:\langle f^{-1}\rangle(I\sqcap^{\mathfrak{Z}_1}J)=\langle f^{-1}\rangle I\sqcap\langle f^{-1}\rangle J.$$

Proof. $\mathfrak{A}$ and $\mathfrak{B}$ are complete lattices (corollary 4.107).

$(\mathfrak{B};\mathfrak{Z}_1)$ is a filtrator with separable core by the theorem 4.112.

$(\mathfrak{B};\mathfrak{Z}_1)$ is finitely meet-closed by the theorem 4.97.

$\mathfrak{A}$ and $\mathfrak{B}$ are starrish by corollary 4.114.

$\mathfrak{A}$ is separable by obvious 4.136.

We are under conditions of the theorem 15.25.

⇒. Obvious (taking into account that $(\mathfrak{B};\mathfrak{Z}_1)$ is finitely meet-closed).

⇐. $\langle f^{-1}\rangle(\mathcal{I}\sqcap\mathcal{J})=\bigsqcap\langle\langle f^{-1}\rangle\rangle\operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)}(\mathcal{I}\sqcap\mathcal{J})=\bigsqcap\langle\langle f^{-1}\rangle\rangle\{I\sqcap^{\mathfrak{Z}_1}J\mid I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}\}=\bigsqcap\{\langle f^{-1}\rangle(I\sqcap^{\mathfrak{Z}_1}J)\mid I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}\}=\bigsqcap\{\langle f^{-1}\rangle I\sqcap\langle f^{-1}\rangle J\mid I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}\}=\bigsqcap\{\langle f^{-1}\rangle I\mid I\in\operatorname{up}\mathcal{I}\}\sqcap\bigsqcap\{\langle f^{-1}\rangle J\mid J\in\operatorname{up}\mathcal{J}\}=\langle f^{-1}\rangle\mathcal{I}\sqcap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{J}$ (used theorem 15.25, theorem 4.110, theorem 15.15). □

15.14 Elements closed regarding a pointfree funcoid

Let $\mathfrak{A}$ be a poset. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{A})$.

Definition 15.106. Let's call *closed* regarding a pointfree funcoid f such element $a\in\mathfrak{A}$ that $\langle f\rangle a\sqsubseteq a$.

Proposition 15.107. If i and j are closed (regarding a pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{A})$), S is a set of closed elements (regarding f), then

1. $i\sqcup j$ is a closed element, if $\mathfrak{A}$ is a separable starrish join-semilattice;
2. $\bigsqcap S$ is a closed element if $\mathfrak{A}$ is a separable complete lattice.

Proof. $\langle f\rangle(i\sqcup j)=\langle f\rangle i\sqcup\langle f\rangle j\sqsubseteq i\sqcup j$ (theorem 15.15), $\langle f\rangle\bigsqcap S\sqsubseteq\bigsqcap\langle\langle f\rangle\rangle S\sqsubseteq\bigsqcap S$ (used separability of $\mathfrak{A}$ twice). Consequently the elements $i\sqcup j$ and $\bigsqcap S$ are closed. □

Proposition 15.108. If S is a set of elements closed regarding a complete pointfree funcoid f with separable destination which is a complete lattice, then the element $\bigsqcup S$ is also closed regarding our funcoid.

Proof. $\langle f\rangle\bigsqcup S=\bigsqcup\langle\langle f\rangle\rangle S\sqsubseteq\bigsqcup S$. □

15.15 Connectedness regarding a pointfree funcoid

Let $\mathfrak{A}$ be a poset with least element. Let $\mu\in\mathsf{FCD}(\mathfrak{A};\mathfrak{A})$. [TODO: No necessity for least element.]

Definition 15.109. An element $a\in\mathfrak{A}$ is called *connected* regarding a pointfree funcoid μ over $\mathfrak{A}$ when

$$\forall x,y\in\mathfrak{A}\setminus\{0^{\mathfrak{A}}\}:(x\sqcup y=a\Rightarrow x\,[\mu]\,y).$$