Proposition 15.101. Monovalued pointfree funcoids between sets of filters on boolean lattices are metamonovalued.[FIXME: Monovalued is defined below.]

Proof. $\langle (\bigsqcap G) \circ f \rangle x = \langle \bigsqcap G \rangle \langle f \rangle x = \bigsqcap_{g \in G} \langle g \rangle \langle f \rangle x = \bigsqcap_{g \in G} \langle g \circ f \rangle x = \langle \bigsqcap_{g \in G} (g \circ f) \rangle x$ for every ultrafilter $x \in \operatorname{atoms}^{\operatorname{Src} f}$. Thus $(\bigsqcap G) \circ f = \bigsqcap_{g \in G} (g \circ f)$. □

15.13 Monovalued and injective pointfree funcoids

Definition 15.102. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. Let $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

The pointfree funcoid f is:

- *monovalued* when $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}$.
- *injective* when $f^{-1} \circ f \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{A})}$.

Monovaluedness is dual of injectivity.

Proposition 15.103. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. Let $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

The pointfree funcoid f is:

- monovalued iff $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}_{\operatorname{im} f}$, if $\operatorname{im} f$ is defined and $\mathfrak{B}$ is a meet-semilattice;
- injective iff $f^{-1} \circ f \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{A})}_{\operatorname{dom} f}$, if $\operatorname{dom} f$ is defined and $\mathfrak{A}$ is a meet-semilattice.

Proof. It's enough to prove $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})} \Leftrightarrow f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}_{\operatorname{im} f}$.

⇐. Obvious.

⇒. Let $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}$. Then $\langle f \circ f^{-1} \rangle x \sqsubseteq x$; and $\langle f \circ f^{-1} \rangle x \sqsubseteq \operatorname{im} f$. Thus $\langle f \circ f^{-1} \rangle x \sqsubseteq x \sqcap \operatorname{im} f = \left\langle \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}_{\operatorname{im} f} \right\rangle x$.

$\langle (f \circ f^{-1})^{-1} \rangle x \sqsubseteq x$ and $\langle (f \circ f^{-1})^{-1} \rangle x = \langle f \circ f^{-1} \rangle x \sqsubseteq \operatorname{im} f$. Thus $\langle (f \circ f^{-1})^{-1} \rangle x \sqsubseteq x \sqcap \operatorname{im} f = \left\langle \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}_{\operatorname{im} f} \right\rangle x$.

Thus $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}_{\operatorname{im} f}$. □

Theorem 15.104. Let $\mathfrak{A}$ be an atomistic meet-semilattice with least element, $\mathfrak{B}$ be an atomistic bounded meet-semilattice. The following statements are equivalent for every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$:

1. f is monovalued.
2. $\forall a \in \operatorname{atoms}^{\mathfrak{A}}: \langle f \rangle a \in \operatorname{atoms}^{\mathfrak{B}} \cup \{0^{\mathfrak{B}}\}$.
3. $\forall i, j \in \mathfrak{A}: \langle f^{-1} \rangle (i \sqcap j) = \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j$.

Proof.

(2)⇒(3). Let $a \in \operatorname{atoms}^{\mathfrak{A}}$, $\langle f \rangle a = b$. Then because $b \in \operatorname{atoms}^{\mathfrak{B}} \cup \{0^{\mathfrak{B}}\}$

$$(i \sqcap j) \sqcap b \neq 0^{\mathfrak{B}} \Leftrightarrow i \sqcap b \neq 0^{\mathfrak{B}} \wedge j \sqcap b \neq 0^{\mathfrak{B}};$$
$$a \,[f]\, i \sqcap j \Leftrightarrow a \,[f]\, i \wedge a \,[f]\, j;$$
$$i \sqcap j \,[f^{-1}]\, a \Leftrightarrow i \,[f^{-1}]\, a \wedge j \,[f^{-1}]\, a;$$
$$a \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle (i \sqcap j) \neq 0^{\mathfrak{A}} \Leftrightarrow a \sqcap \langle f^{-1} \rangle i \neq 0^{\mathfrak{A}} \wedge a \sqcap \langle f^{-1} \rangle j \neq 0^{\mathfrak{A}};$$
$$a \sqcap \langle f^{-1} \rangle (i \sqcap j) \neq 0^{\mathfrak{A}} \Leftrightarrow a \sqcap \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j \neq 0^{\mathfrak{A}};$$
$$\langle f^{-1} \rangle (i \sqcap j) = \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j.$$

(3)⇒(1). $\langle f^{-1} \rangle a \sqcap \langle f^{-1} \rangle b = \langle f^{-1} \rangle (a \sqcap b) = \langle f^{-1} \rangle 0^{\mathfrak{B}} = 0^{\mathfrak{A}}$ (by proposition 15.13 because $\mathfrak{A}$ is separable by proposition 3.22) for every two distinct $a, b \in \operatorname{atoms}^{\mathfrak{B}}$. This is equivalent to $\neg (\langle f^{-1} \rangle a \ [f]\ b)$; $b \sqcap \langle f \rangle \langle f^{-1} \rangle a = 0^{\mathfrak{B}}$; $b \sqcap \langle f \circ f^{-1} \rangle a = 0^{\mathfrak{B}}$; $\neg (a \ [f \circ f^{-1}]\ b)$. So $a \,[f \circ f^{-1}]\, b \Rightarrow a = b$ for every $a, b \in \operatorname{atoms}^{\mathfrak{B}}$. This is possible only (corollary 15.56 and the fact that $\mathfrak{B}$ is atomic) when $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{FCD}(\mathfrak{B})}$.