Definition 15.89. Let $\mathfrak{Z}_0$ and $\mathfrak{Z}_1$ be join-semilattices with least elements. I will call *pointfree generalized closure* such a function $\alpha \in (\mathfrak{Z}_1)^{\mathfrak{Z}_0}$ that [TODO: It is just a map preserving finite joins, no need to introduce a new term. It can be generalized for arbitrary posets.]

1. $\alpha 0^{\mathfrak{Z}_0} = 0^{\mathfrak{Z}_1}$;
2. $\forall I, J \in \mathfrak{Z}_0 : \alpha(I \sqcup J) = \alpha I \sqcup \alpha J$.

Definition 15.90. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. I will call a *co-complete pointfree funcoid* a pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ such that $\langle f \rangle|_{\mathfrak{Z}_0}$ is a pointfree generalized closure.

Proposition 15.91. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Co-complete pointfree funcoids $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ bijectively correspond to pointfree generalized closures $\mathfrak{Z}_1^{\mathfrak{Z}_0}$, where the bijection is $f \mapsto \langle f \rangle|_{\mathfrak{Z}_0}$.

Proof. It follows from the theorem 15.26. □

Theorem 15.92. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ be semifiltered, star-separable, down-aligned filtrator with finitely meet closed, join-closed, and separable core, where $\mathfrak{Z}_0$ is a complete boolean lattice and both $\mathfrak{Z}_0$ and $\mathfrak{A}$ are atomistic lattices.

Let $(\mathfrak{B}; \mathfrak{Z}_1)$ be a star-separable filtrator.

The following conditions are equivalent for every pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$:

1. f^{-1} is co-complete;
2. $\forall S \in \mathscr{P}\mathfrak{A}, J \in \mathfrak{Z}_1 : (\bigsqcup^{\mathfrak{A}} S \, [f] \, J \Rightarrow \exists \mathcal{I} \in S : \mathcal{I} \, [f] \, J)$;
3. $\forall S \in \mathscr{P}\mathfrak{Z}_0, J \in \mathfrak{Z}_1 : (\bigsqcup^{\mathfrak{Z}_0} S \, [f] \, J \Rightarrow \exists I \in S : I \, [f] \, J)$;
4. f is complete;
5. $\forall S \in \mathscr{P}\mathfrak{Z}_0 : \langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S = \bigsqcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S$.

Proof. First note that the theorem 4.53 applies to the filtrator $(\mathfrak{A}; \mathfrak{Z}_0)$.

(3)⇒(1). For every $S \in \mathscr{P}\mathfrak{Z}_0$, $J \in \mathfrak{Z}_1$

$$\bigsqcup^{\mathfrak{Z}_0} S \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle J \neq 0^{\mathfrak{A}} \Rightarrow \exists I \in S : I \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle J \neq 0^{\mathfrak{A}}, \tag{15.9}$$

consequently by the theorem 4.53 we have $\langle f^{-1} \rangle J \in \mathfrak{Z}_0$.

(1)⇒(2). For every $S \in \mathscr{P}\mathfrak{A}$, $J \in \mathfrak{Z}_1$ we have $\langle f^{-1} \rangle J \in \mathfrak{Z}_0$, consequently

$$\forall S \in \mathscr{P}\mathfrak{A}, J \in \mathfrak{Z}_1 : \left(\bigsqcup^{\mathfrak{A}} S \not\asymp \langle f^{-1} \rangle J \Rightarrow \exists \mathcal{I} \in S : \mathcal{I} \not\asymp \langle f^{-1} \rangle J \right).$$

From this follows (2).

(2)⇒(4). Let $\langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S$ and $\bigsqcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S$ be defined. We have $\langle f \rangle \bigsqcup^{\mathfrak{A}} S = \langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S$. $J \sqcap^{\mathfrak{B}} \langle f \rangle \bigsqcup^{\mathfrak{A}} S \neq 0^{\mathfrak{B}} \Leftrightarrow \bigsqcup^{\mathfrak{A}} S \, [f] \, J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I} \, [f] \, J \Leftrightarrow \exists \mathcal{I} \in S : J \sqcap^{\mathfrak{B}} \langle f \rangle \mathcal{I} \neq 0^{\mathfrak{B}} \Leftrightarrow J \sqcap^{\mathfrak{B}} \bigsqcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S \neq 0^{\mathfrak{B}}$ (used theorem 4.53). Thus $\langle f \rangle \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S$ by star-separability of $(\mathfrak{B}; \mathfrak{Z}_1)$.

(5)⇒(3). Let $\langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S$ be defined. Then $\bigsqcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S$ is also defined because $\langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S = \bigsqcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S$. Then $\bigsqcup^{\mathfrak{Z}_0} S \, [f] \, J \Leftrightarrow J \sqcap^{\mathfrak{B}} \langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S \neq 0^{\mathfrak{B}} \Leftrightarrow J \sqcap^{\mathfrak{B}} \bigsqcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S \neq 0^{\mathfrak{B}}$ what by theorem 4.53 is equivalent to $\exists I \in S : J \sqcap^{\mathfrak{B}} \langle f \rangle I \neq 0^{\mathfrak{B}}$ that is $\exists I \in S : I \, [f] \, J$.

(2)⇒(3), (4)⇒(5). By join-closedness of the core of $(\mathfrak{A}; \mathfrak{Z}_0)$. □

Theorem 15.93. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. If R is a set of co-complete pointfree funcoids in $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ then $\bigsqcup R$ is a co-complete pointfree funcoid.