2. $\operatorname{atoms}(f \sqcup \bigsqcap R) = \operatorname{atoms} f \cup \operatorname{atoms} \bigsqcap R = \operatorname{atoms} f \cup \bigcap \langle \operatorname{atoms} \rangle R = \bigcap \langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R = \bigcap \langle \operatorname{atoms} \rangle \langle f \sqcup \rangle R = \operatorname{atoms} \bigsqcap \langle f \sqcup \rangle R$. (Used the following equality.)

$$\begin{aligned} \langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R &= \\ \{ (\operatorname{atoms} f) \cup A \mid A \in \langle \operatorname{atoms} \rangle R \} &= \\ \{ (\operatorname{atoms} f) \cup A \mid \exists C \in R : A = \operatorname{atoms} C \} &= \\ \{ (\operatorname{atoms} f) \cup (\operatorname{atoms} C) \mid C \in R \} &= \\ \{ \operatorname{atoms}(f \sqcup C) \mid C \in R \} &= \\ \{ \operatorname{atoms} B \mid \exists C \in R : B = f \sqcup C \} &= \\ \{ \operatorname{atoms} B \mid B \in \langle f \sqcup \rangle R \} &= \\ \langle \operatorname{atoms} \rangle \langle f \sqcup \rangle. & \end{aligned}$$

□

Corollary 15.84. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is a co-brouwerian lattice.

Proposition 15.85. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be sets of filters over some boolean lattices and $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$, $g \in \mathsf{FCD}(\mathfrak{B}; \mathfrak{C})$. Let $\mathfrak{B}$ be an atomic poset. Then

$\operatorname{atoms}(g \circ f) = \{ x \times^{\mathsf{FCD}} z \mid x \in \operatorname{atoms}^{\mathfrak{A}}, z \in \operatorname{atoms}^{\mathfrak{C}}, \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms} g) \}$.

Proof. $(x \times^{\mathsf{FCD}} z) \sqcap (g \circ f) \neq 0^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{C})} \Leftrightarrow x \, [g \circ f] \, z \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \, [f] \, y \wedge y \, [g] \, z) \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((x \times^{\mathsf{FCD}} y) \sqcap f \neq 0^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} \wedge (y \times^{\mathsf{FCD}} z) \sqcap g \neq 0^{\mathsf{FCD}(\mathfrak{B}; \mathfrak{C})})$ (were used corollary 15.69 and theorem 15.61). □

Theorem 15.86. Let f be a pointfree funcoid between sets of filters on boolean lattices.

1. $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$;
2. $\langle f \rangle \mathcal{X} = \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$.

Proof. 1. $\exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y} \Leftrightarrow \exists a \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Dst} f)} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge \mathcal{X} \, [a \times^{\mathsf{FCD}} b] \, \mathcal{Y}) \Leftrightarrow \exists a \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathfrak{F}(\operatorname{Dst} f)} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge a \times^{\mathsf{FCD}} b \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \Leftrightarrow \exists F \in \operatorname{atoms} f : (F \not\asymp f \wedge F \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \Leftrightarrow f \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} \Leftrightarrow \mathcal{X} \, [f] \, \mathcal{Y}$.

2. Let $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$. Suppose $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$. Then $\mathcal{X} \, [f] \, \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{Y} \not\asymp \langle F \rangle \mathcal{X}$; $\mathcal{Y} \not\asymp \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. So $\langle f \rangle \mathcal{X} \sqsubseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. The contrary $\langle f \rangle \mathcal{X} \sqsupseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ is obvious. □

15.11 Complete pointfree funcoids

Definition 15.87. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is *complete*, when for every $S \in \mathscr{P}\mathfrak{A}$ whenever both $\bigsqcup S$ and $\bigsqcup \langle \langle f \rangle \rangle S$ are defined we have

$$\langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle S.$$

Proposition 15.88. Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is complete iff $\langle f \rangle a = \bigsqcup \langle \langle f \rangle \rangle \operatorname{atoms} a$ for every $a \in \mathfrak{A}$.

Proof. Direct implication is obvious. The reverse implication:

Let S be a set of filters.

$\langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle \operatorname{atoms} \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle \bigcup \langle \operatorname{atoms} \rangle S = \bigsqcup \bigcup \langle \langle \langle f \rangle \rangle \rangle \langle \operatorname{atoms} \rangle S = \bigsqcup \langle \bigsqcup \rangle \langle \langle \langle f \rangle \rangle \rangle \langle \operatorname{atoms} \rangle S = \bigsqcup \langle \bigsqcup \circ \langle \langle f \rangle \rangle \circ \operatorname{atoms} \rangle S = \bigsqcup \{ \bigsqcup \langle \langle f \rangle \rangle \operatorname{atoms} a \mid a \in S \} = \bigsqcup \{ \langle f \rangle a \mid a \in S \} = \bigsqcup \langle \langle f \rangle \rangle S$. □