Theorem 15.73. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ be primary filtrators over boolean lattices. If $\mathcal{A}\in\mathfrak{A}$ then $\mathcal{A}\times^{\mathsf{FCD}}$ is a complete homomorphism of the lattice $\mathfrak{A}$ to a the lattice $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, if also $\mathcal{A}\neq 0^{\mathfrak{A}}$ then it is an order embedding.

Proof. Let $S\in\mathscr{P}\mathfrak{A}$, $X\in\mathfrak{Z}_0$, $x\in\operatorname{atoms}^{\mathfrak{A}}$.

$$\begin{aligned}\left\langle\bigsqcup\langle\mathcal{A}\times^{\mathsf{FCD}}\rangle S\right\rangle X &= \bigsqcup\{\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle X \mid \mathcal{B}\in S\}\\ &= \begin{cases}\bigsqcup S & \text{if } X\sqcap^{\mathfrak{A}}\mathcal{A}\neq 0^{\mathfrak{A}}\\ 0^{\mathfrak{B}} & \text{if } X\sqcap^{\mathfrak{A}}\mathcal{A}=0^{\mathfrak{A}}\end{cases}\\ &= \left\langle\mathcal{A}\times^{\mathsf{FCD}}\bigsqcup S\right\rangle X.\end{aligned}$$

Thus $\bigsqcup\langle\mathcal{A}\times^{\mathsf{FCD}}\rangle S=\mathcal{A}\times^{\mathsf{FCD}}\bigsqcup S$ by theorem 15.25.

$$\begin{aligned}\left\langle\bigsqcap\langle\mathcal{A}\times^{\mathsf{FCD}}\rangle S\right\rangle x &= \bigsqcap\{\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle x \mid \mathcal{B}\in S\}\\ &= \begin{cases}\bigsqcap S & \text{if } x\sqcap^{\mathfrak{A}}\mathcal{A}\neq 0^{\mathfrak{A}}\\ 0^{\mathfrak{B}} & \text{if } x\sqcap^{\mathfrak{A}}\mathcal{A}=0^{\mathfrak{A}}\end{cases}\\ &= \left\langle\mathcal{A}\times^{\mathsf{FCD}}\bigsqcap S\right\rangle x.\end{aligned}$$

Thus $\bigsqcap\langle\mathcal{A}\times^{\mathsf{FCD}}\rangle S=\mathcal{A}\times^{\mathsf{FCD}}\bigsqcap S$ by theorem 15.54.

If $\mathcal{A}\neq 0^{\mathfrak{A}}$ then obviously the function $\mathcal{A}\times^{\mathsf{FCD}}$ is injective. □

Proposition 15.74. Let $\mathfrak{A}$ be a meet-semilattice with least element and $\mathfrak{B}$ be a poset with least element. If a is an atom of $\mathfrak{A}$, $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ then $f|_a{=}a\times^{\mathsf{FCD}}\langle f\rangle a$.

Proof. Let $\mathcal{X}\in\mathfrak{A}$.

$$\mathcal{X}\sqcap a\neq 0^{\mathfrak{A}}\Rightarrow\langle f|_a\rangle\mathcal{X}=\langle f\rangle a,\quad \mathcal{X}\sqcap a=0^{\mathfrak{A}}\Rightarrow\langle f|_a\rangle\mathcal{X}=0^{\mathfrak{B}}.$$ □

Proposition 15.75. $f\circ(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})=\mathcal{A}\times^{\mathsf{FCD}}\langle f\rangle\mathcal{B}$ for elements $\mathcal{A}\in\mathfrak{A}$ and $B\in\mathfrak{B}$ of some posets $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ with least elements and $f\in\mathsf{FCD}(\mathfrak{B};\mathfrak{C})$.

Proof. Let $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{B}$.

$\langle f\circ(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\rangle\mathcal{X}=\left(\begin{cases}\langle f\rangle\mathcal{B} & \text{if } \mathcal{X}\not\asymp\mathcal{A}\\ 0 & \text{if } \mathcal{X}\asymp\mathcal{A}\end{cases}\right)=\langle\mathcal{A}\times^{\mathsf{FCD}}\langle f\rangle\mathcal{B}\rangle\mathcal{X}$.

$\langle(f\circ(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}))^{-1}\rangle\mathcal{Y}=\langle(\mathcal{B}\times^{\mathsf{FCD}}\mathcal{A})\circ f^{-1}\rangle\mathcal{Y}=\left(\begin{cases}\mathcal{A} & \text{if } \langle f^{-1}\rangle\mathcal{Y}\not\asymp\mathcal{B}\\ 0 & \text{if } \langle f^{-1}\rangle\mathcal{Y}\asymp\mathcal{B}\end{cases}\right)=\left(\begin{cases}\mathcal{A} & \text{if } \mathcal{Y}\not\asymp\langle f\rangle\mathcal{B}\\ 0 & \text{if } \mathcal{Y}\asymp\langle f\rangle\mathcal{B}\end{cases}\right)=$ $\langle\langle f\rangle\mathcal{B}\times^{\mathsf{FCD}}\mathcal{A}\rangle\mathcal{Y}=\langle(\mathcal{A}\times^{\mathsf{FCD}}\langle f\rangle\mathcal{B})^{-1}\rangle\mathcal{Y}$. □

15.10 Atomic pointfree funcoids

Theorem 15.76. Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. A $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is an atom of the poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ iff there exist $a\in\operatorname{atoms}^{\mathfrak{A}}$ and $b\in\operatorname{atoms}^{\mathfrak{B}}$ such that $f=a\times^{\mathsf{FCD}}b$.

Proof. $\mathfrak{A}$ and $\mathfrak{B}$ are atomic by the theorem 4.135.

- **⇒.** Let f be an atom of the poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$. Let's get elements $a\in\operatorname{atoms}\operatorname{dom}f$ and $b\in\operatorname{atoms}\langle f\rangle a$. Then for every $\mathcal{X}\in\mathfrak{A}$

  $$\mathcal{X}\asymp^{\mathfrak{A}}a\Rightarrow\langle a\times^{\mathsf{FCD}}b\rangle\mathcal{X}=0^{\mathfrak{B}}\sqsubseteq\langle f\rangle\mathcal{X},\quad \mathcal{X}\not\asymp^{\mathfrak{A}}a\Rightarrow\langle a\times^{\mathsf{FCD}}b\rangle\mathcal{X}=b\sqsubseteq\langle f\rangle\mathcal{X}.$$

  So $a\times^{\mathsf{FCD}}b\sqsubseteq f$; because f is atomic we have $f=a\times^{\mathsf{FCD}}b$.

- **⇐.** Let $a\in\operatorname{atoms}^{\mathfrak{A}}$, $b\in\operatorname{atoms}^{\mathfrak{B}}$, $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$. If $b\asymp^{\mathfrak{B}}\langle f\rangle a$ then $\neg(a\,[f]\,b)$, $f\sqcap(a\times^{\mathsf{FCD}}b)=0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ (because $\mathfrak{A}$ and $\mathfrak{B}$ are bounded meet-semilattices); if $b\sqsubseteq\langle f\rangle a$ then $\forall\mathcal{X}\in\mathfrak{A}$: $(\mathcal{X}\not\asymp a\Rightarrow\langle f\rangle\mathcal{X}\sqsupseteq b)$, $f\sqsupseteq a\times^{\mathsf{FCD}}b$. Consequently $f\sqcap(a\times^{\mathsf{FCD}}b)=0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\vee f\sqsupseteq a\times^{\mathsf{FCD}}b$; that is $a\times^{\mathsf{FCD}}b$ is an atomic pointfree funcoid. □