From this, as easy to show, $h \sqsubseteq f$ and $h \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. If $g \sqsubseteq f \wedge g \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a $g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ then $\operatorname{dom} g \sqsubseteq \mathcal{A}$, $\operatorname{im} g \sqsubseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \sqcap \langle g \rangle (\mathcal{A} \sqcap \mathcal{X}) \sqsubseteq \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \mathcal{X}) = \left\langle \mathrm{id}_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle \mathrm{id}_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

and similarly $\langle g^{-1} \rangle \mathcal{X} \sqsubseteq \langle h^{-1} \rangle \mathcal{X}$.

$g \sqsubseteq h$. So $h = f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

Corollary 15.68. Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. For every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$ we have $f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{B}})$.

Proof. $f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{B}}) = \mathrm{id}_{1^{\mathfrak{B}}}^{\mathsf{FCD}(\mathfrak{B})} \circ f \circ \mathrm{id}_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} = f \circ \mathrm{id}_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} = f|_{\mathcal{A}}$. □

Corollary 15.69. Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. For every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$ we have

$$f \not\asymp \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \mathcal{A} \, [f] \, \mathcal{B}.$$

Proof. $f \not\asymp \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \Leftrightarrow \left\langle f \sqcap (\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}) \right\rangle 1^{\mathfrak{A}} \neq 0^{\mathfrak{B}} \Leftrightarrow \left\langle \mathrm{id}_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \circ f \circ \mathrm{id}_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle 1^{\mathfrak{A}} \neq 0^{\mathfrak{B}} \Leftrightarrow \left\langle \mathrm{id}_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle \mathrm{id}_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})} \right\rangle 1^{\mathfrak{A}} \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap 1^{\mathfrak{A}}) \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{B} \sqcap \langle f \rangle \mathcal{A} \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{A} \, [f] \, \mathcal{B}$. □

Theorem 15.70. Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. Then the poset $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is separable.

Proof. Let $f, g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $f \neq g$. By the theorem 15.12 $[f] \neq [g]$. That is there exist $x, y \in \mathfrak{A}$ such that $x \, [f] \, y \nLeftrightarrow x \, [g] \, y$ that is $f \sqcap (x \times^{\mathsf{FCD}} y) \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \nLeftrightarrow g \sqcap (x \times^{\mathsf{FCD}} y) \neq 0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$. Thus $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is separable. □

Theorem 15.71. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets of filters over boolean lattices. If $S \in \mathscr{P}(\mathfrak{A} \times \mathfrak{B})$ then

$$\bigsqcap \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} = \bigsqcap \operatorname{dom} S \times^{\mathsf{FCD}} \bigsqcap \operatorname{im} S.$$

Proof. If $x \in \operatorname{atoms}^{\mathfrak{A}}$ then by the theorem 15.59

$$\left\langle \bigsqcap \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \right\rangle x = \bigsqcap \{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \}.$$

If $x \sqcap \bigsqcap \operatorname{dom} S \neq 0^{\mathfrak{A}}$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S: (x \sqcap \mathcal{A} \neq 0^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \mathcal{B});$$
$$\{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} = \operatorname{im} S;$$

if $x \sqcap \bigsqcap \operatorname{dom} S = 0^{\mathfrak{A}}$ then

$$\exists (\mathcal{A}; \mathcal{B}) \in S: (x \sqcap \mathcal{A} = 0^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = 0^{\mathfrak{B}});$$
$$\{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} \ni 0^{\mathfrak{B}}.$$

So

$$\left\langle \bigsqcap \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \right\rangle x = \begin{cases} \bigsqcap \operatorname{im} S & \text{if } x \sqcap \bigsqcap \operatorname{dom} S \neq 0^{\mathfrak{A}}; \\ 0^{\mathfrak{B}} & \text{if } x \sqcap \bigsqcap \operatorname{dom} S = 0^{\mathfrak{A}}. \end{cases}$$

From this by theorem 15.58 the statement of our theorem follows. □

Corollary 15.72. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets of filters over boolean lattices.

For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{A}$ and $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{B}$

$$(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1).$$

Proof. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = \bigsqcap \{ \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \}$ what is by the last theorem equal to $(\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1)$. □