2. Let denote $x\,\delta\,y \Leftrightarrow \forall f\in R: x\,[f]\,y$.

$$\begin{gathered}\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a, Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X, y\in\mathrm{atoms}^{\mathfrak{B}}Y: x\,\delta\,y\Rightarrow\\ \forall f\in R, X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a, Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\mathrm{atoms}^{\mathfrak{A}}X, y\in\mathrm{atoms}^{\mathfrak{B}}Y: x\,[f]\,y\Rightarrow\\ \forall f\in R, X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a, Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b: X\,[f]\,Y\Rightarrow\\ \forall f\in R: a\,[f]\,b\Leftrightarrow\\ a\,\delta\,b.\end{gathered}$$

So, by the theorem 15.58, δ can be continued till $[p]$ for some $p\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

For every $q\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ such that $\forall f\in R: q\sqsubseteq f$ we have $x\,[q]\,y\Rightarrow\forall f\in R: x\,[f]\,y\Leftrightarrow x\,\delta\,y\Leftrightarrow x\,[p]\,y$, so $q\sqsubseteq p$. Consequently $p=\bigsqcap R$.

From this $x\,[\bigsqcap R]\,y\Leftrightarrow\forall f\in R: x\,[f]\,y$.

1. From the former $y\in\mathrm{atoms}^{\mathfrak{B}}\langle\bigsqcap R\rangle x\Leftrightarrow y\sqcap\langle\bigsqcap R\rangle x\neq 0^{\mathfrak{B}}\Leftrightarrow\forall f\in R: y\sqcap\langle f\rangle x\neq 0^{\mathfrak{B}}\Leftrightarrow y\in\bigcap\langle\mathrm{atoms}^{\mathfrak{B}}\rangle\{\langle f\rangle x\mid f\in R\}\Leftrightarrow y\in\mathrm{atoms}\bigsqcap\{\langle f\rangle x\mid f\in R\}$ for every $y\in\mathrm{atoms}^{\mathfrak{B}}$.

$\mathfrak{B}$ is atomically separable by the corollary 4.138. Thus

$$\left\langle\bigsqcap R\right\rangle x=\bigsqcap\{\langle f\rangle x\mid f\in R\}.$$ □

15.8 More on composition of pointfree funcoids

Proposition 15.60. $[g\circ f]=[g]\circ\langle f\rangle=\langle g^{-1}\rangle^{-1}\circ[f]$ for every composable pointfree funcoids f and g.

Proof. $x\,[g\circ f]\,y\Leftrightarrow y\not\asymp\langle g\circ f\rangle x\Leftrightarrow y\not\asymp\langle g\rangle\langle f\rangle x\Leftrightarrow\langle f\rangle x\,[g]\,y\Leftrightarrow x([g]\circ\langle f\rangle)y$ for every $x\in\mathfrak{A}$, $y\in\mathfrak{B}$. Thus $[g\circ f]=[g]\circ\langle f\rangle$. $[g\circ f]=[(f^{-1}\circ g^{-1})^{-1}]=[f^{-1}\circ g^{-1}]^{-1}=([f^{-1}]\circ\langle g^{-1}\rangle)^{-1}=\langle g^{-1}\rangle^{-1}\circ[f]$. □

Theorem 15.61. Let f and g be pointfree funcoids and $\mathfrak{A}=\mathrm{Dst}\,f=\mathrm{Src}\,g$ is an atomic poset. Then for every $\mathcal{X}\in\mathrm{Src}\,f$ and $\mathcal{Z}\in\mathrm{Dst}\,g$

$$\mathcal{X}\,[g\circ f]\,\mathcal{Z}\Leftrightarrow\exists y\in\mathrm{atoms}^{\mathfrak{A}}: (\mathcal{X}\,[f]\,y\wedge y\,[g]\,\mathcal{Z}).$$

Proof.

$$\begin{aligned}\exists y\in\mathrm{atoms}^{\mathfrak{A}}: (\mathcal{X}\,[f]\,y\wedge y\,[g]\,\mathcal{Z}) &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{A}}: (\mathcal{Z}\not\asymp\langle g\rangle y\wedge y\not\asymp\langle f\rangle\mathcal{X})\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{A}}: (y\not\asymp\langle g^{-1}\rangle\mathcal{Z}\wedge y\not\asymp\langle f\rangle\mathcal{X})\\ &\Leftrightarrow \langle g^{-1}\rangle\mathcal{Z}\not\asymp\langle f\rangle\mathcal{X}\\ &\Leftrightarrow \mathcal{X}\,[g\circ f]\,\mathcal{Z}.\end{aligned}$$

□

Theorem 15.62. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be separable starrish join-semilattices and $\mathfrak{B}$ is atomic. Then:

1. $f\circ(g\sqcup h)=f\circ g\sqcup f\circ h$ for $g,h\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $f\in\mathsf{FCD}(\mathfrak{B};\mathfrak{C})$.
2. $(g\sqcup h)\circ f=g\circ f\sqcup h\circ f$ for $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $g,h\in\mathsf{FCD}(\mathfrak{B};\mathfrak{C})$.

Proof. I will prove only the first equality because the other is analogous.

We can apply theorem 15.35.

For every $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{C}$

$$\begin{aligned}\mathcal{X}\,[f\circ(g\sqcup h)]\,\mathcal{Z} &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{B}}: (\mathcal{X}\,[g\sqcup h]\,y\wedge y\,[f]\,\mathcal{Z})\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{B}}: ((\mathcal{X}\,[g]\,y\vee\mathcal{X}\,[h]\,y)\wedge y\,[f]\,\mathcal{Z})\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{B}}: ((\mathcal{X}\,[g]\,y\wedge y\,[f]\,\mathcal{Z})\vee(\mathcal{X}\,[h]\,y\wedge y\,[f]\,\mathcal{Z}))\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{B}}: (\mathcal{X}\,[g]\,y\wedge y\,[f]\,\mathcal{Z})\vee\exists y\in\mathrm{atoms}^{\mathfrak{B}}: (\mathcal{X}\,[h]\,y\wedge y\,[f]\,\mathcal{Z})\\ &\Leftrightarrow \mathcal{X}\,[f\circ g]\,\mathcal{Z}\vee\mathcal{X}\,[f\circ h]\,\mathcal{Z}\\ &\Leftrightarrow \mathcal{X}\,[f\circ g\sqcup f\circ h]\,\mathcal{Z}.\end{aligned}$$