15.7 Specifying funcoids by functions or relations on atomic filters

Theorem 15.54. Let $\mathfrak{A}$ be an atomic poset and $(\mathfrak{B}; \mathfrak{Z}_1)$ is a primary filtrator over a boolean lattice. Then for every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{X} \in \mathfrak{A}$ we have

$$\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}.$$

Proof. For every $Y \in \mathfrak{Z}_1$ we have

$$\begin{aligned} Y \not\asymp^{\mathfrak{B}} \langle f \rangle \mathcal{X} &\Leftrightarrow \mathcal{X} \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \\ &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}: x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \\ &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}: Y \not\asymp^{\mathfrak{B}} \langle f \rangle x. \end{aligned}$$

Thus $\partial \langle f \rangle \mathcal{X} = \bigcup \langle \partial \rangle \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} = \partial \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ (used theorem 4.132). Consequently $\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ by the corollary 4.128. □

Proposition 15.55. Let f be a pointfree funcoid. Then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

1. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}: x [f] \mathcal{Y}$ if $\operatorname{Src} f$ is an atomic poset.
2. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists y \in \operatorname{atoms} \mathcal{Y}: \mathcal{X} [f] y$ if $\operatorname{Dst} f$ is an atomic poset.

Proof. I will prove only the second as the first is similar.

If $\mathcal{X} [f] \mathcal{Y}$, then $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$, consequently exists $y \in \operatorname{atoms} \mathcal{Y}$ such that $y \not\asymp \langle f \rangle \mathcal{X}$, $\mathcal{X} [f] y$. The reverse is obvious. □

Corollary 15.56. If f is a pointfree funcoid with both source and destination being atomic posets, then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y}: x [f] y.$$

Proof. Apply the theorem twice. □

Corollary 15.57. If $\mathfrak{A}$ is a separable atomic poset and $\mathfrak{B}$ is a separable poset then $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is determined by the values of $\langle f \rangle X$ for $X \in \operatorname{atoms}^{\mathfrak{A}}$.

Proof. $y \not\asymp \langle f \rangle x \Leftrightarrow x \not\asymp \langle f^{-1} \rangle y \Leftrightarrow \exists X \in \operatorname{atoms} x: X \not\asymp \langle f^{-1} \rangle y \Leftrightarrow \exists X \in \operatorname{atoms} x: y \not\asymp \langle f \rangle X$.

Thus by separability of $\mathfrak{B}$ we have $\langle f \rangle$ is determined by $\langle f \rangle X$ for $X \in \operatorname{atoms} x$.

By separability of $\mathfrak{A}$ we infer that f can be restored from $\langle f \rangle$ (theorem 15.12). □

Theorem 15.58. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ be primary filtrators over boolean lattices.

1. A function $\alpha \in \mathfrak{B}^{\operatorname{atoms}^{\mathfrak{A}}}$ such that (for every $a \in \operatorname{atoms}^{\mathfrak{A}}$)

$$\alpha a \sqsubseteq \bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle \circ \operatorname{atoms}^{\mathfrak{A}} \right\rangle \operatorname{up}^{(\mathfrak{A}; \mathfrak{Z}_0)} a \tag{15.5}$$

can be continued to the function $\langle f \rangle$ for a unique $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$;

$$\langle f \rangle \mathcal{X} = \bigsqcup \langle \alpha \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} \tag{15.6}$$

for every $\mathcal{X} \in \mathfrak{A}$.

2. A relation $\delta \in \mathscr{P}(\operatorname{atoms}^{\mathfrak{A}} \times \operatorname{atoms}^{\mathfrak{B}})$ such that (for every $a \in \operatorname{atoms}^{\mathfrak{A}}$, $b \in \operatorname{atoms}^{\mathfrak{B}}$)

$$\forall X \in \operatorname{up}^{(\mathfrak{A}; \mathfrak{Z}_0)} a, Y \in \operatorname{up}^{(\mathfrak{B}; \mathfrak{Z}_1)} b \, \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y: x \, \delta \, y \Rightarrow a \, \delta \, b \tag{15.7}$$

can be continued to the relation $[f]$ for a unique $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$;

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y}: x \, \delta \, y \tag{15.8}$$

for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$.