Proof. For every $y \in \operatorname{Dst} f$ we have $y \not\asymp \langle f\rangle (x \sqcap \operatorname{dom} f) \Leftrightarrow x \sqcap \operatorname{dom} f \sqcap \langle f^{-1}\rangle y \neq 0^{\operatorname{Src} f} \Leftrightarrow x \sqcap \operatorname{im} f^{-1} \sqcap \langle f^{-1}\rangle y \neq 0^{\operatorname{Src} f} \Leftrightarrow x \sqcap \langle f^{-1}\rangle y \neq 0^{\operatorname{Src} f} \Leftrightarrow y \not\asymp \langle f\rangle x$. Thus $\langle f\rangle x = \langle f\rangle (x \sqcap \operatorname{dom} f)$ by separability of $\operatorname{Dst} f$. □

Proposition 15.50. $x \not\asymp \operatorname{dom} f \Leftrightarrow (\langle f\rangle x$ is not least) for every pointfree funcoid f and $x \in \operatorname{Src} f$ if $\operatorname{Dst} f$ has greatest element 1 and $\operatorname{Src} f$ is a separable poset.

Proof. $x \not\asymp \operatorname{dom} f \Leftrightarrow x \not\asymp \langle f^{-1}\rangle 1^{\operatorname{Dst} f} \Leftrightarrow 1^{\operatorname{Dst} f} \not\asymp \langle f\rangle x \Leftrightarrow (\langle f\rangle x$ is not least). □

Corollary 15.51. $\operatorname{dom} f = \bigsqcup \{a \in \operatorname{atoms}^{\operatorname{Src} f} \mid \langle f\rangle a \neq 0^{\operatorname{Dst} f}\}$ for every pointfree funcoid f whose destination is a bounded poset and source is a separable atomistic meet-semilattice.

Proof. For every $a \in \operatorname{atoms}^{\operatorname{Src} f}$ we have $a \not\asymp \operatorname{dom} f \Leftrightarrow a \not\asymp \langle f^{-1}\rangle 1^{\operatorname{Dst} f} \Leftrightarrow 1^{\operatorname{Dst} f} \not\asymp \langle f\rangle a \Leftrightarrow \langle f\rangle a \neq 0^{\operatorname{Dst} f}$. So

$$\operatorname{dom} f = \bigsqcup \{a \in \operatorname{atoms}^{\operatorname{Src} f} \mid a \not\asymp \operatorname{dom} f\} = \bigsqcup \{a \in \operatorname{atoms}^{\operatorname{Src} f} \mid \langle f\rangle a \neq 0^{\operatorname{Dst} f}\}.$$ □

Proposition 15.52. $\operatorname{dom}(f|_a) = a \sqcap \operatorname{dom} f$ for every pointfree funcoid f and $a \in \operatorname{Src} f$ where $\operatorname{Src} f$ is a separable meet-semilattice and $\operatorname{Dst} f$ has greatest element.

Proof. $\operatorname{dom}(f|_a) = \operatorname{im}\left(\operatorname{id}_a^{\mathsf{FCD}(\operatorname{Src} f)} \circ f^{-1}\right) = \left\langle \operatorname{id}_a^{\mathsf{FCD}(\operatorname{Src} f)}\right\rangle \langle f^{-1}\rangle 1^{\operatorname{Dst} f} = a \sqcap \langle f^{-1}\rangle 1^{\operatorname{Dst} f} = a \sqcap \operatorname{dom} f$. □

Proposition 15.53. For every composable pointfree funcoids f and g where the posets $\operatorname{Src} f$ and $\operatorname{Dst} f = \operatorname{Src} g$ have greatest elements and $\operatorname{Dst} f$ and $\operatorname{Dst} g$ are separable:

1. If $\operatorname{im} f \sqsupseteq \operatorname{dom} g$ then $\operatorname{im}(g \circ f) = \operatorname{im} g$.
2. If $\operatorname{im} f \sqsubseteq \operatorname{dom} g$ then $\operatorname{dom}(g \circ f) = \operatorname{dom} g$.

Proof.

1. $\operatorname{im}(g \circ f) = \langle g \circ f\rangle 1^{\operatorname{Src} f} = \langle g\rangle \langle f\rangle 1^{\operatorname{Src} f} = \langle g\rangle \operatorname{im} f = \langle g\rangle \operatorname{dom} g = \langle g\rangle 1^{\operatorname{Src} g} = \operatorname{im} g$.
2. $\operatorname{dom}(g \circ f) = \operatorname{im}(f^{-1} \circ g^{-1})$ what by the proved is equal to $\operatorname{im} f^{-1}$ that is $\operatorname{dom} f$. □

15.6 Category of pointfree funcoids

I will define the category $p\mathsf{FCD}$ of pointfree funcoids:

- The class of objects are small posets.
- The set of morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.
- The composition is the composition of pointfree funcoids.
- Identity morphism for an object $\mathfrak{A}$ is $(\mathfrak{A}; \mathfrak{A}; \operatorname{id}_{\mathfrak{A}}; \operatorname{id}_{\mathfrak{A}})$.

To prove that it is really a category is trivial.

The *category of pointfree funcoid triples* is defined as follows:

- Objects are pairs $(\mathfrak{A}; \mathcal{A})$ where $\mathfrak{A}$ is a small poset and $\mathcal{A} \in \mathfrak{A}$.
- The morphisms from an object $(\mathfrak{A}; \mathcal{A})$ to an object $(\mathfrak{B}; \mathcal{B})$ are tuples $(\mathfrak{A}; \mathfrak{B}; \mathcal{A}; \mathcal{B}; f)$ where $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$. [FIXME: Domain and image are not always defined. Even if it's defined, the composition law may not hold. We can require instead $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$, but this is defined only for posets with least elements.]
- The composition is defined by the formula $(\mathcal{B}; \mathcal{C}; g) \circ (\mathcal{A}; \mathcal{B}; f) = (\mathcal{A}; \mathcal{C}; g \circ f)$.
- Identity morphism for an object $(\mathfrak{A}; \mathcal{A})$ is $\operatorname{id}_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}$. [FIXME: Defined only for meet-semilattices. We can also define a wider precategory without identity.]

To prove that it is really a category is trivial.