2. $x [f \sqcup g] y \Leftrightarrow y \not\asymp \langle f \sqcup g \rangle x \Leftrightarrow y \not\asymp \langle f \rangle x \sqcup \langle g \rangle x \Leftrightarrow y \not\asymp \langle f \rangle x \vee y \not\asymp \langle g \rangle x \Leftrightarrow x [f] y \vee x [g] y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. □

Theorem 15.36. Let f be a pointfree funcoid from a separable poset $\mathfrak{A}$ to a separable poset $\mathfrak{B}$. If $\langle f \rangle$ is an injection, then $\langle f \rangle$ is an order embedding $\mathfrak{A} \rightarrow \mathfrak{B}$.

Proof. Suppose $x \sqsupseteq y$ but $\langle f \rangle x \not\sqsupseteq \langle f \rangle y$.

Then by separability of $\mathfrak{B}$ there exist $z \not\asymp \langle f \rangle y$ such that $z \asymp \langle f \rangle x$.

Thus $\langle f^{-1} \rangle z \asymp x$ and $\langle f^{-1} \rangle z \not\asymp y$ what is impossible for $x \sqsupseteq y$. □

Corollary 15.37. Let f be a pointfree funcoid from a separable poset $\mathfrak{A}$ to a separable poset $\mathfrak{B}$. If $\langle f \rangle$ is a bijection $\mathfrak{A} \rightarrow \mathfrak{B}$, then $\langle f \rangle$ is an order isomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$.

15.5 Domain and range of a pointfree funcoid

Definition 15.38. Let $\mathfrak{A}$ be a poset. The *identity pointfree funcoid* $\mathrm{id}^{\mathsf{FCD}(\mathfrak{A})} = (\mathfrak{A}; \mathfrak{A}; \mathrm{id}_{\mathfrak{A}}; \mathrm{id}_{\mathfrak{A}})$.

It is trivial that identity funcoid is really a pointfree funcoid.

Let now $\mathfrak{A}$ be a meet-semilattice.

Definition 15.39. Let $a \in \mathfrak{A}$. The *restricted identity pointfree funcoid* $\mathrm{id}_a^{\mathsf{FCD}(\mathfrak{A})} = (\mathfrak{A}; \mathfrak{A}; a \sqcap^{\mathfrak{A}}; a \sqcap^{\mathfrak{A}})$.

Proposition 15.40. The restricted pointfree funcoid is a pointfree funcoid.

Proof. We need to prove that $(a \sqcap^{\mathfrak{A}} x) \not\asymp^{\mathfrak{A}} y \Leftrightarrow (a \sqcap^{\mathfrak{A}} y) \not\asymp^{\mathfrak{A}} x$ what is obvious. □

Obvious 15.41. $\left(\mathrm{id}_a^{\mathsf{FCD}(\mathfrak{A})}\right)^{-1} = \mathrm{id}_a^{\mathsf{FCD}(\mathfrak{A})}$.

Obvious 15.42. $x \left[\mathrm{id}_a^{\mathsf{FCD}(\mathfrak{A})}\right] y \Leftrightarrow a \not\asymp^{\mathfrak{A}} x \sqcap^{\mathfrak{A}} y$ for every $x, y \in \mathfrak{A}$.

Definition 15.43. I will define *restricting* of a pointfree funcoid f to an element $a \in \mathrm{Src}\, f$ by the formula $f|_a \overset{\mathrm{def}}{=} f \circ \mathrm{id}_a^{\mathsf{FCD}(\mathrm{Src}\, f)}$.

Definition 15.44. *Image* of f will be defined by the formula $\mathrm{im}\, f = \bigsqcup \langle \langle f \rangle \rangle^* \mathrm{Src}\, f$.

Obvious 15.45. $\mathrm{im}\, f \sqsupseteq fx$ for every $x \in \mathrm{Src}\, f$ whenever $\mathrm{im}\, f$ is defined.

Proposition 15.46. $\mathrm{im}\, f = \langle f \rangle 1$ if $\mathrm{Src}\, f$ has greatest element 1 and $\mathrm{Dst}\, f$ is a separable poset.

Proof. $\langle f \rangle 1$ is greater than every $\langle f \rangle x$ (where $x \in \mathrm{Src}\, f$) by proposition 15.14 and thus

$$\langle f \rangle 1 = \max \langle \langle f \rangle \rangle^* \mathrm{Src}\, f = \mathrm{im}\, f.$$ □

Definition 15.47. *Domain* of a pointfree funcoid f is defined by the formula $\mathrm{dom}\, f = \mathrm{im} f^{-1}$.

Proposition 15.48. $\langle f \rangle \mathrm{dom}\, f = \mathrm{im}\, f$ if f is a pointfree funcoid and $\mathrm{Src}\, f$ has greatest element 1 and $\mathrm{Dst}\, f$ is a separable poset.

Proof. $y \not\asymp \langle f \rangle \mathrm{dom}\, f \Leftrightarrow \mathrm{dom}\, f \not\asymp \langle f^{-1} \rangle y \Leftrightarrow \langle f^{-1} \rangle 1 \not\asymp \langle f^{-1} \rangle y \Leftrightarrow \langle f^{-1} \rangle y \neq 0 \Leftrightarrow 1 \not\asymp \langle f^{-1} \rangle y \Leftrightarrow y \not\asymp \langle f \rangle 1 \Leftrightarrow y \not\asymp \mathrm{im}\, f$ for every $y \in \mathrm{Dst}\, f$.

So $\langle f \rangle \mathrm{dom}\, f = \mathrm{im}\, f$ by separability of $\mathrm{Dst}\, f$. □

Proposition 15.49. $\langle f \rangle x = \langle f \rangle (x \sqcap \mathrm{dom}\, f)$ whenever $\mathrm{dom}\, f$ is defined, for every $x \in \mathrm{Src}\, f$ for a pointfree funcoid f whose source is a meet-semilattice with least element and destination is a separable poset with least element.