Transitivity. It follows from transitivity of the order relations on $\mathfrak{A}$ and $\mathfrak{B}$.

Antisymmetry. It follows from antisymmetry of the order relations on $\mathfrak{A}$ and $\mathfrak{B}$. □

Remark 15.30. It is enough to define order of pointfree funcoids on every set $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ where $\mathfrak{A}$ and $\mathfrak{B}$ are posets. We do not need to compare pointfree funcoids with different sources or destinations.

Obvious 15.31. $f \sqsubseteq g \Rightarrow [f] \subseteq [g]$ for every $f, g \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ for every posets $\mathfrak{A}$ and $\mathfrak{B}$.

Theorem 15.32. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable posets then $f \sqsubseteq g \Leftrightarrow [f] \subseteq [g]$.

Proof. From the theorem 15.12. □

Theorem 15.33. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then for $R \in \mathscr{P}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $X \in \mathfrak{Z}_0$, $Y \in \mathfrak{Z}_1$ we have:

1. $X\,[\bigsqcup R]\,Y \Leftrightarrow \exists f \in R: X\,[f]\,Y$;
2. $\langle \bigsqcup R \rangle X = \bigsqcup \{ \langle f \rangle X \mid f \in R \}$.

Proof.

2. $\alpha X \overset{\text{def}}{=} \bigsqcup \{ \langle f \rangle X \mid f \in R \}$ (by corollary 4.107 all joins on $\mathfrak{B}$ exist). We have $\alpha 0^{\mathfrak{A}} = 0^{\mathfrak{B}}$;

$$\begin{aligned} \alpha(I \sqcup^{\mathfrak{Z}_0} J) &= \bigsqcup \{ \langle f \rangle (I \sqcup^{\mathfrak{Z}_0} J) \mid f \in R \} \\ &= \bigsqcup \{ \langle f \rangle (I \sqcup^{\mathfrak{A}} J) \mid f \in R \} \\ &= \bigsqcup \{ \langle f \rangle I \sqcup^{\mathfrak{B}} \langle f \rangle J \mid f \in R \} \\ &= \bigsqcup \{ \langle f \rangle I \mid f \in R \} \sqcup^{\mathfrak{B}} \bigsqcup \{ \langle f \rangle J \mid f \in R \} \\ &= \alpha I \sqcup^{\mathfrak{B}} \alpha J \end{aligned}$$

(used theorem 15.15). By theorem 15.26 the function α can be continued to $\langle h \rangle$ for an $h \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$. Obviously

$$\forall f \in R: h \sqsupseteq f. \tag{15.4}$$

And h is the least element of $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ for which the condition (15.4) holds. So $h = \bigsqcup R$.

1. $X\,[\bigsqcup R]\,Y \Leftrightarrow Y \sqcap^{\mathfrak{B}} \langle \bigsqcup R \rangle X \neq 0^{\mathfrak{B}} \Leftrightarrow Y \sqcap^{\mathfrak{B}} \bigsqcup \{ \langle f \rangle X \mid f \in R \} \neq 0^{\mathfrak{B}} \Leftrightarrow \exists f \in R: Y \sqcap^{\mathfrak{B}} \langle f \rangle X \neq 0^{\mathfrak{B}} \Leftrightarrow \exists f \in R: X\,[f]\,Y$ (used theorem 4.118). □

Corollary 15.34. If $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices then $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a complete lattice.

Proof. Apply [26]. □

Theorem 15.35. Let $\mathfrak{A}$ and $\mathfrak{B}$ be starrish join-semilattices. Then for $f, g \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$:

1. $\langle f \sqcup g \rangle x = \langle f \rangle x \sqcup \langle g \rangle x$ for every $x \in \mathfrak{A}$;
2. $[f \sqcup g] = [f] \cup [g]$.

Proof.

1. Let $\alpha \mathcal{X} \overset{\text{def}}{=} \langle f \rangle x \sqcup \langle g \rangle x$; $\beta \mathcal{Y} \overset{\text{def}}{=} \langle f^{-1} \rangle y \sqcup \langle g^{-1} \rangle y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. Then

$$\begin{aligned} y \not\asymp \alpha x &\Leftrightarrow y \not\asymp \langle f \rangle x \vee y \not\asymp \langle g \rangle x \\ &\Leftrightarrow x \not\asymp \langle f^{-1} \rangle y \vee x \not\asymp \langle g^{-1} \rangle y \\ &\Leftrightarrow x \not\asymp \langle f^{-1} \rangle y \sqcup \langle g^{-1} \rangle y \\ &\Leftrightarrow x \not\asymp \beta y. \end{aligned}$$

So $h = (\mathfrak{A};\mathfrak{B};\alpha;\beta)$ is a pointfree funcoid. Obviously $h \sqsupseteq f$ and $h \sqsupseteq g$. If $p \sqsupseteq f$ and $p \sqsupseteq g$ for some $p \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ then $\langle p \rangle x \sqsupseteq \langle f \rangle x \sqcup \langle g \rangle x = \langle h \rangle x$ and $\langle p^{-1} \rangle y \sqsupseteq \langle f^{-1} \rangle y \sqcup \langle g^{-1} \rangle y = \langle h^{-1} \rangle y$ that is $p \sqsupseteq h$. So $f \sqcup g = h$.