By properties of generalized filter bases, $\bigsqcap \langle \mathcal{Y} \sqcap \rangle \langle \alpha \rangle \mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)} \mathcal{X} \neq 0^{\mathfrak{B}}$ is equivalent to

$$\forall X \in \mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)} \mathcal{X}: \mathcal{Y} \sqcap \alpha X \neq 0^{\mathfrak{B}},$$

what is equivalent to $\forall X \in \mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)} \mathcal{X}, Y \in \mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)} \mathcal{Y}: Y \sqcap^{\mathfrak{B}} \alpha X \neq 0^{\mathfrak{B}} \Leftrightarrow \forall X \in \mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)} \mathcal{X}, Y \in \mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)} \mathcal{Y}: Y \in \partial(\alpha X) \Leftrightarrow \forall X \in \mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)} \mathcal{X}, Y \in \mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)} \mathcal{Y}: X \,\delta\, Y$. Combining the equivalencies we get $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{X} \,\delta'\, \mathcal{Y}$. Analogously $\mathcal{X} \sqcap \beta' \mathcal{Y} \neq 0^{\mathfrak{A}} \Leftrightarrow \mathcal{X} \,\delta'\, \mathcal{Y}$. So $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{X} \sqcap \beta' \mathcal{Y} \neq 0^{\mathfrak{A}}$, that is $(\mathfrak{A}; \mathfrak{B}; \alpha'; \beta')$ is a pointfree funcoid. From the formula $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq 0^{\mathfrak{B}} \Leftrightarrow \mathcal{X} \,\delta'\, \mathcal{Y}$ it follows that $[(\mathfrak{A}; \mathfrak{B}; \alpha'; \beta')]$ is a continuation of δ.

1. Let define the relation $\delta \in \mathscr{P}(\mathfrak{Z}_0 \times \mathfrak{Z}_1)$ by the formula $X \,\delta\, Y \Leftrightarrow Y \sqcap^{\mathfrak{B}} \alpha X \neq 0^{\mathfrak{B}}$.

That $\neg(0^{\mathfrak{Z}_0} \,\delta\, I')$ and $\neg(I \,\delta\, 0^{\mathfrak{Z}_1})$ is obvious. We have $K \,\delta\, I' \sqcup^{\mathfrak{Z}_1} J' \Leftrightarrow (I' \sqcup^{\mathfrak{Z}_1} J') \sqcap^{\mathfrak{B}} \alpha K \neq 0^{\mathfrak{B}} \Leftrightarrow (I' \sqcup^{\mathfrak{B}} J') \sqcap \alpha K \neq 0^{\mathfrak{B}} \Leftrightarrow (I' \sqcap^{\mathfrak{B}} \alpha K) \sqcup (J' \sqcap^{\mathfrak{B}} \alpha K) \neq 0^{\mathfrak{B}} \Leftrightarrow I' \sqcap^{\mathfrak{B}} \alpha K \neq 0^{\mathfrak{B}} \vee J' \sqcap^{\mathfrak{B}} \alpha K \neq 0^{\mathfrak{B}} \Leftrightarrow K \,\delta\, I' \vee K \,\delta\, J'$ and $I \sqcup^{\mathfrak{Z}_0} J \,\delta\, K' \Leftrightarrow K' \sqcap^{\mathfrak{B}} \alpha(I \sqcup^{\mathfrak{Z}_0} J) \neq 0^{\mathfrak{B}} \Leftrightarrow K' \sqcap^{\mathfrak{B}} (\alpha I \sqcup \alpha J) \neq 0^{\mathfrak{B}} \Leftrightarrow (K' \sqcap^{\mathfrak{B}} \alpha I) \sqcup (K' \sqcap^{\mathfrak{B}} \alpha J) \neq 0^{\mathfrak{B}} \Leftrightarrow K' \sqcap^{\mathfrak{B}} \alpha I \neq 0^{\mathfrak{B}} \vee K' \sqcap^{\mathfrak{B}} \alpha J \neq 0^{\mathfrak{B}} \Leftrightarrow I \,\delta\, K' \vee J \,\delta\, K'$.

That is the formulas (15.2) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

$\forall X \in \mathfrak{Z}_0, Y \in \mathfrak{Z}_1: (Y \sqcap^{\mathfrak{B}} \langle f \rangle X \neq 0^{\mathfrak{B}} \Leftrightarrow X \,[f]\, Y \Leftrightarrow Y \sqcap^{\mathfrak{B}} \alpha X \neq 0^{\mathfrak{B}})$, consequently $\forall X \in \mathfrak{Z}_0: \alpha X = \langle f \rangle X$ because our filtrator is with separable core. So $\langle f \rangle$ is a continuation of α. □

Proposition 15.27. Let $(\mathrm{Src}\, f; \mathfrak{Z}_0)$ be a primary filtrator over a bounded distributive lattice and $(\mathrm{Dst}\, f; \mathfrak{Z}_1)$ is a primary filtrator over a boolean lattice. If S is a generalized filter base on $\mathrm{Src}\, f$ then $\langle f \rangle \bigsqcap^{\mathrm{Src}\, f} S = \bigsqcap^{\mathrm{Dst}\, f} \langle \langle f \rangle \rangle S$ for every pointfree funcoid f.

Proof. First the meets $\bigsqcap^{\mathrm{Src}\, f} S$ and $\bigsqcap^{\mathrm{Dst}\, f} \langle \langle f \rangle \rangle S$ exist by corollary 4.107.

$(\mathrm{Src}\, f; \mathfrak{Z}_0)$ is a finitely meet-closed filtrator by proposition 4.97 and with separable core by theorem 4.112; thus we can apply theorem 15.25 ($\mathrm{up}\, x \neq \emptyset$ is obvious).

$\langle f \rangle \bigsqcap^{\mathrm{Src}\, f} S \sqsubseteq \langle f \rangle X$ for every $X \in S$ because $\mathrm{Dst}\, f$ is separable by obvious 4.136 and thus $\langle f \rangle \bigsqcap^{\mathrm{Src}\, f} S \sqsubseteq \bigsqcap^{\mathrm{Dst}\, f} \langle \langle f \rangle \rangle S$.

Taking into account properties of generalized filter bases:

$$\begin{aligned}
\langle f \rangle \bigsqcap^{\mathrm{Src}\, f} S &= \\
\bigsqcap^{\mathrm{Dst}\, f} \langle \langle f \rangle \rangle \mathrm{up} \bigsqcap S &= \\
\bigsqcap^{\mathrm{Dst}\, f} \langle \langle f \rangle \rangle \{ X \mid \exists \mathcal{P} \in S: X \in \mathrm{up}\, \mathcal{P} \} &= \\
\bigsqcap^{\mathrm{Dst}\, f} \{ \langle f \rangle X \mid \exists \mathcal{P} \in S: X \in \mathrm{up}\, \mathcal{P} \} &\sqsupseteq \text{ (because Dst } f \text{ is a separable poset)} \\
\bigsqcap^{\mathrm{Dst}\, f} \{ \langle f \rangle \mathcal{P} \mid \mathcal{P} \in S \} &= \\
\bigsqcap^{\mathrm{Dst}\, f} \langle \langle f \rangle \rangle S.&
\end{aligned}$$

□

15.4 The order of pointfree funcoids

Definition 15.28. The order of pointfree funcoids $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is defined by the formula:

$$f \sqsubseteq g \Leftrightarrow \forall x \in \mathfrak{A}: \langle f \rangle x \sqsubseteq \langle g \rangle x \wedge \forall y \in \mathfrak{B}: \langle f^{-1} \rangle y \sqsubseteq \langle g^{-1} \rangle y.$$

Proposition 15.29. It is really a partial order on the set $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

Proof.

Reflexivity. Obvious.