## 15.2 Composition of pointfree funcoids

**Definition 15.17.** *Composition* of pointfree funcoids is defined by the formula

$$(\mathfrak{B}; \mathfrak{C}; \alpha_2; \beta_2) \circ (\mathfrak{A}; \mathfrak{B}; \alpha_1; \beta_1) = (\mathfrak{A}; \mathfrak{C}; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

**Definition 15.18.** I will call funcoids $f$ and $g$ *composable* when $\operatorname{Dst} f = \operatorname{Src} g$.

**Proposition 15.19.** If $f$, $g$ are composable pointfree funcoids then $g \circ f$ is pointfree funcoid.

**Proof.** Let $f = (\mathfrak{A}; \mathfrak{B}; \alpha_1; \beta_1)$, $g = (\mathfrak{B}; \mathfrak{C}; \alpha_2; \beta_2)$. For every $x, y \in \mathfrak{A}$ we have

$$y \not\asymp (\alpha_2 \circ \alpha_1) x \Leftrightarrow y \not\asymp \alpha_2 \alpha_1 x \Leftrightarrow \alpha_1 x \not\asymp \beta_2 y \Leftrightarrow x \not\asymp \beta_1 \beta_2 y \Leftrightarrow x \not\asymp (\beta_1 \circ \beta_2) y.$$

So $(\mathfrak{A}; \mathfrak{C}; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)$ is a pointfree funcoid. □

**Obvious 15.20.** $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$ for every composable pointfree funcoids $f$ and $g$.

**Theorem 15.21.** $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every composable pointfree funcoids $f$ and $g$.

**Proof.**

$$\langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle = \langle f^{-1} \circ g^{-1} \rangle;$$
$$\langle ((g \circ f)^{-1})^{-1} \rangle = \langle g \circ f \rangle = \langle (f^{-1} \circ g^{-1})^{-1} \rangle.$$

□

**Proposition 15.22.** $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable pointfree funcoids $f$, $g$, $h$.

**Proof.**
$\langle (h \circ g) \circ f \rangle = \langle h \circ g \rangle \circ \langle f \rangle = \langle h \rangle \circ \langle g \rangle \circ \langle f \rangle = \langle h \rangle \circ \langle g \circ f \rangle = \langle h \circ (g \circ f) \rangle$.
$\langle ((h \circ g) \circ f)^{-1} \rangle = \langle f^{-1} \circ (h \circ g)^{-1} \rangle = \langle f^{-1} \circ g^{-1} \circ h^{-1} \rangle = \langle (g \circ f)^{-1} \circ h^{-1} \rangle = \langle (h \circ (g \circ f))^{-1} \rangle$. □

## 15.3 Pointfree funcoid as continuation

**Proposition 15.23.** Let $f$ be a pointfree funcoid. Then for every $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$ we have

1. If $(\operatorname{Src} f; \mathfrak{Z})$ is a filtrator with separable core then $x \, [f] \, y \Leftrightarrow \forall X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z})} x: X \, [f] \, y$.
2. If $(\operatorname{Dst} f; \mathfrak{Z})$ is a filtrator with separable core then $x \, [f] \, y \Leftrightarrow \forall Y \in \operatorname{up}^{(\operatorname{Dst} f; \mathfrak{Z})} y: x \, [f] \, Y$.

**Proof.** We will prove only the second because the first is similar.
$x \, [f] \, y \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow \forall Y \in \operatorname{up}^{(\operatorname{Dst} f; \mathfrak{Z})} y: Y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow \forall Y \in \operatorname{up}^{(\operatorname{Dst} f; \mathfrak{Z})} y: x \, [f] \, Y$. □

**Corollary 15.24.** Let $f$ be a pointfree funcoid and $(\operatorname{Src} f; \mathfrak{Z}_0)$, $(\operatorname{Dst} f; \mathfrak{Z}_1)$ are filtrators with separable core. Then

$$x \, [f] \, y \Leftrightarrow \forall X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x, Y \in \operatorname{up}^{(\operatorname{Dst} f; \mathfrak{Z}_1)} y: X \, [f] \, Y.$$

**Proof.** Apply the proposition twice. □

**Theorem 15.25.** Let $f$ be a pointfree funcoid. Let $(\operatorname{Src} f; \mathfrak{Z}_0)$ be a finitely meet-closed filtrator with separable core which is a meet-semilattice and $\forall x \in \operatorname{Src} f: \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x \neq \emptyset$ and $(\operatorname{Dst} f; \mathfrak{Z}_1)$ is a primary filtrator over a boolean lattice.

$$\langle f \rangle x = \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x.$$

**Proof.** By the previous proposition for every $y \in \operatorname{Dst} f$:

$$y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow x \, [f] \, y \Leftrightarrow \forall X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x: X \, [f] \, y \Leftrightarrow \forall X \in \operatorname{up}^{(\operatorname{Src} f; \mathfrak{Z}_0)} x: y \not\asymp^{\operatorname{Dst} f} \langle f \rangle X.$$