Obvious 15.10. $x\,[f]\,y \Leftrightarrow y \not\asymp \langle f\rangle x \Leftrightarrow x \not\asymp \langle f^{-1}\rangle y$ for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$.

Obvious 15.11. $[f^{-1}] = [f]^{-1}$ for every pointfree funcoid f.

Theorem 15.12. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. Then:

1. If $\mathfrak{A}$ is separable, for given value of $\langle f\rangle$ there exists no more than one $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.
2. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable, for given value of $[f]$ there exists no more than one $f \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

Proof. Let $f, g \in \mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

1. Let $\langle f\rangle = \langle g\rangle$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have $x \not\asymp \langle f^{-1}\rangle y \Leftrightarrow y \not\asymp \langle f\rangle x \Leftrightarrow y \not\asymp \langle g\rangle x \Leftrightarrow x \not\asymp \langle g^{-1}\rangle y$ and thus by separability of $\mathfrak{A}$ we have $\langle f^{-1}\rangle y = \langle g^{-1}\rangle y$ that is $\langle f^{-1}\rangle = \langle g^{-1}\rangle$ and so $f = g$.
2. Let $[f] = [g]$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have $y \not\asymp \langle f\rangle x \Leftrightarrow x\,[f]\,y \Leftrightarrow x\,[g]\,y \Leftrightarrow y \not\asymp \langle g\rangle x$ and thus by separability of $\mathfrak{B}$ we have $\langle f\rangle x = \langle g\rangle x$ that is $\langle f\rangle = \langle g\rangle$. Similarly we have $\langle f^{-1}\rangle = \langle g^{-1}\rangle$. Thus $f = g$. □

Proposition 15.13. If $\operatorname{Src} f$ and $\operatorname{Dst} f$ have least elements, then $\langle f\rangle 0^{\operatorname{Src} f} = 0^{\operatorname{Dst} f}$ for every pointfree funcoid f. [TODO: Previously I required separability of Dst f. It turned out to be a superfluous condition. Remove it also in consequences of this proposition.]

Proof. $y \not\asymp \langle f\rangle 0^{\operatorname{Src} f} \Leftrightarrow 0^{\operatorname{Src} f} \not\asymp \langle f^{-1}\rangle y \Leftrightarrow 0$ for every $y \in \operatorname{Dst} f$. Thus $\langle f\rangle 0^{\operatorname{Src} f} \asymp \langle f\rangle 0^{\operatorname{Src} f}$. So $\langle f\rangle 0^{\operatorname{Src} f} = 0^{\operatorname{Dst} f}$. □

Proposition 15.14. If $\operatorname{Dst} f$ is a separable meet-semilattice then $\langle f\rangle$ is a monotone function (for a pointfree funcoid f). [FIXME: Added condition to be a meet-semilattice, add it also in all consequences.] [TODO: Check whether existence of least element of Dst f is required (in theorem 3.14).]

Proof. $a \sqsubseteq b \Rightarrow \forall x \in \operatorname{Dst} f\colon (a \not\asymp \langle f^{-1}\rangle x \Rightarrow b \not\asymp \langle f^{-1}\rangle x) \Rightarrow \forall x \in \operatorname{Dst} f\colon (x \not\asymp \langle f\rangle a \Rightarrow x \not\asymp \langle f\rangle b) \Leftrightarrow \star\langle f\rangle a \subseteq \star\langle f\rangle b \Rightarrow \langle f\rangle a \sqsubseteq \langle f\rangle b$ (used theorem 3.14 and that it is a separable meet-semilattice). □

Theorem 15.15. Let f be a pointfree funcoid from a starrish join-semilattice $\operatorname{Src} f$ to a separable starrish join-semilattice $\operatorname{Dst} f$. Then $\langle f\rangle(i \sqcup j) = \langle f\rangle i \sqcup \langle f\rangle j$ for every $i, j \in \operatorname{Src} f$.

Proof.

$$\begin{aligned}
\star\langle f\rangle(i \sqcup j) &= \\
\{y \in \operatorname{Dst} f \mid y \not\asymp \langle f\rangle(i \sqcup j)\} &= \\
\{y \in \operatorname{Dst} f \mid i \sqcup j \not\asymp \langle f^{-1}\rangle y\} &= \\
\{y \in \operatorname{Dst} f \mid i \not\asymp \langle f^{-1}\rangle y \vee j \not\asymp \langle f^{-1}\rangle y\} &= \\
\{y \in \operatorname{Dst} f \mid y \not\asymp \langle f\rangle i \vee y \not\asymp \langle f\rangle j\} &= \\
\{y \in \operatorname{Dst} f \mid y \not\asymp \langle f\rangle i \sqcup \langle f\rangle j\} &= \\
\star(\langle f\rangle i \sqcup \langle f\rangle j).
\end{aligned}$$

Thus $\langle f\rangle(i \sqcup j) = \langle f\rangle i \sqcup \langle f\rangle j$ by separability. □

Proposition 15.16. Let f be a pointfree funcoid. Then:

1. $k\,[f]\,i \sqcup j \Leftrightarrow k\,[f]\,i \vee k\,[f]\,j$ for every $i, j \in \operatorname{Dst} f$, $k \in \operatorname{Src} f$ if $\operatorname{Dst} f$ is a starrish join-semilattice.
2. $i \sqcup j\,[f]\,k \Leftrightarrow i\,[f]\,k \vee j\,[f]\,k$ for every $i, j \in \operatorname{Src} f$, $k \in \operatorname{Dst} f$ if $\operatorname{Src} f$ is a starrish join-semilattice.

Proof. 1. $k\,[f]\,i \sqcup j \Leftrightarrow i \sqcup j \not\asymp \langle f\rangle k \Leftrightarrow i \not\asymp \langle f\rangle k \vee j \not\asymp \langle f\rangle k \Leftrightarrow k\,[f]\,i \vee k\,[f]\,j$.

2. Similar. □