Chapter 15
Pointfree funcoids

This chapter is based on [28].

This is a routine chapter. There is almost nothing creative here. I just generalize theorems about funcoids to the maximum extent for *pointfree funcoids* (defined below) preserving the proof idea. The main idea behind this chapter is to find weakest theorem conditions enough for the same theorem statement as for above theorems for funcoids.

For these who know pointfree topology: Pointfree topology notions of frames and locales is a non-trivial generalization of topological spaces. Pointfree funcoids are different: I just replace the set of filters on a set with an arbitrary poset, this readily gives the definition of *pointfree funcoid*, almost no need of creativity here.

Pointfree funcoids are used in the below definitions of products of funcoids.

15.1 Definition

Definition 15.1. *Pointfree funcoid* is a quadruple $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ where $\mathfrak{A}$ and $\mathfrak{B}$ are posets, $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ and $\beta \in \mathfrak{A}^{\mathfrak{B}}$ such that

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B}: (y \not\asymp \alpha x \Leftrightarrow x \not\asymp \beta y).$$

Definition 15.2. The *source* $\operatorname{Src}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{A}$ and *destination* $\operatorname{Dst}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{B}$ for every pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

To every funcoid $(A; B; \alpha; \beta)$ corresponds pointfree funcoid $(\mathscr{P}A; \mathscr{P}B; \alpha; \beta)$. Thus pointfree funcoids are a generalization of funcoids.

Definition 15.3. I will denote $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ the set of pointfree funcoids from $\mathfrak{A}$ to $\mathfrak{B}$ (that is with source $\mathfrak{A}$ and destination $\mathfrak{B}$), for every posets $\mathfrak{A}$ and $\mathfrak{B}$.

Proposition 15.4. If $\mathfrak{A}$ and $\mathfrak{B}$ have least elements, then $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ has least element.[TODO: Move below where order of pointfree funcoids is defined.]

Proof. It is $(\mathfrak{A}; \mathfrak{B}; \mathfrak{A} \times \{0^{\mathfrak{B}}\}; \mathfrak{B} \times \{0^{\mathfrak{A}}\})$. □

Definition 15.5. $\langle(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)\rangle \overset{\text{def}}{=} \alpha$ for every pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Definition 15.6. $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)^{-1} = (\mathfrak{B}; \mathfrak{A}; \beta; \alpha)$ for every pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Proposition 15.7. If f is a pointfree funcoid then f^{-1} is also a pointfree funcoid.

Proof. It follows from symmetry in the definition of pointfree funcoid. □

Obvious 15.8. $(f^{-1})^{-1} = f$ for every pointfree funcoid f.

Definition 15.9. The relation $[f] \in \mathscr{P}(\operatorname{Src} f \times \operatorname{Dst} f)$ is defined by the formula (for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$)

$$x\,[f]\,y \overset{\text{def}}{=} y \not\asymp \langle f\rangle x.$$