Chapter 14

Counter-examples about funcoids and reloids

For further examples we will use the filter defined by the formula

$$\Delta = \bigsqcap \left\{ \uparrow^{\mathfrak{F}(\mathbb{R})}(-\varepsilon;\varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \right\}.$$

I will denote $\Omega(A)$ the Fréchet filter on a set A.

Example 14.1. There exist a funcoid f and a set S of funcoids such that $f \sqcap \bigsqcup S \neq \bigsqcup \langle f \sqcap \rangle S$.

Proof. Let $f = \Delta \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}(\mathbb{R})}\{0\}$ and $S = \left\{ \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}((\varepsilon;+\infty) \times \{0\}) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \right\}$. Then $f \sqcap \bigsqcup S = \left(\Delta \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}(\mathbb{R})}\{0\} \right) \sqcap \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}((0;+\infty) \times \{0\}) = \left(\Delta \sqcap \uparrow^{\mathfrak{F}(\mathbb{R})}(0;+\infty) \right) \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}(\mathbb{R})}\{0\} \neq 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$ while $\bigsqcup \langle f \sqcap \rangle S = \bigsqcup \left\{ 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} \right\} = 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$. □

Example 14.2. There exist a set R of funcoids and a funcoid f such that $f \circ \bigsqcup R \neq \bigsqcup \langle f \circ \rangle R$.

Proof. Let $f = \Delta \times^{\mathsf{FCD}} \uparrow^{\mathbb{R}}\{0\}$, $R = \{ \uparrow^{\mathbb{R}}\{0\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{R}}(\varepsilon;+\infty) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \}$.

We have $\bigsqcup R = \uparrow^{\mathbb{R}}\{0\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{R}}(0;+\infty)$; $f \circ \bigsqcup R = \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}(\{0\} \times \{0\}) \neq 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$ and $\bigsqcup \langle f \circ \rangle R = \bigsqcup \left\{ 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} \right\} = 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$. □

Example 14.3. There exist a set R of reloids and a reloid f such that $f \circ \bigsqcup R \neq \bigsqcup \langle f \circ \rangle R$.

Proof. Let $f = \Delta \times^{\mathsf{RLD}} \uparrow^{\mathbb{R}}\{0\}$, $R = \{ \uparrow^{\mathbb{R}}\{0\} \times^{\mathsf{RLD}} \uparrow^{\mathbb{R}}(\varepsilon;+\infty) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \}$.

We have $\bigsqcup R = \uparrow^{\mathbb{R}}\{0\} \times^{\mathsf{RLD}} \uparrow^{\mathbb{R}}(0;+\infty)$; $f \circ \bigsqcup R = \uparrow^{\mathsf{RLD}(\mathbb{R};\mathbb{R})}(\{0\} \times \{0\}) \neq 0^{\mathsf{RLD}(\mathbb{R};\mathbb{R})}$ and $\bigsqcup \langle f \circ \rangle R = \bigsqcup \left\{ 0^{\mathsf{RLD}(\mathbb{R};\mathbb{R})} \right\} = 0^{\mathsf{RLD}(\mathbb{R};\mathbb{R})}$. □

Example 14.4. There exist a set R of funcoids and filters $\mathcal{X}$ and $\mathcal{Y}$ such that

1. $\mathcal{X} \left[\bigsqcup R \right] \mathcal{Y} \wedge \nexists f \in R: \mathcal{X} [f] \mathcal{Y}$;
2. $\langle \bigsqcup R \rangle \mathcal{X} \sqsupset \bigsqcup \{ \langle f \rangle \mathcal{X} \mid f \in R \}$.

Proof.

1. Take $\mathcal{X} = \Delta$ and $\mathcal{Y} = 1^{\mathfrak{F}(\mathbb{R})}$, $R = \left\{ \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}((\varepsilon;+\infty) \times \mathbb{R}) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \right\}$. Then $\bigsqcup R = \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}((0;+\infty) \times \mathbb{R})$. So $\mathcal{X} \left[\bigsqcup R \right] \mathcal{Y}$ and $\forall f \in R: \neg(\mathcal{X} [f] \mathcal{Y})$.
2. With the same $\mathcal{X}$ and R we have $\langle \bigsqcup R \rangle \mathcal{X} = 1^{\mathfrak{F}(\mathbb{R})}$ and $\langle f \rangle \mathcal{X} = 0^{\mathfrak{F}(\mathbb{R})}$ for every $f \in R$, thus $\bigsqcup \{ \langle f \rangle \mathcal{X} \mid f \in R \} = 0^{\mathfrak{F}(\mathbb{R})}$. □

Example 14.5. $\bigsqcup \{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{B} \in T \} \neq \mathcal{A} \times^{\mathsf{RLD}} \bigsqcup T$ for some filter $\mathcal{A}$ and set of filters T (with a common base).

Proof. Take $\mathbb{R}_+ = \{ x \in \mathbb{R} \mid x > 0 \}$, $\mathcal{A} = \Delta$, $T = \{ \uparrow\{x\} \mid x \in \mathbb{R}_+ \}$ where $\uparrow = \uparrow^{\mathbb{R}}$.

$\bigsqcup T = \uparrow\mathbb{R}_+$; $\mathcal{A} \times^{\mathsf{RLD}} \bigsqcup T = \Delta \times^{\mathsf{RLD}} \uparrow\mathbb{R}_+$.

$\bigsqcup \{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{B} \in T \} = \bigsqcup \{ \Delta \times^{\mathsf{RLD}} \uparrow\{x\} \mid x \in \mathbb{R}_+ \}$.

We'll prove that $\bigsqcup \{ \Delta \times^{\mathsf{RLD}} \uparrow\{x\} \mid x \in \mathbb{R}_+ \} \neq \Delta \times^{\mathsf{RLD}} \uparrow\mathbb{R}_+$.

Consider $K = \bigcup \{ \{x\} \times (-1/x; 1/x) \mid x \in \mathbb{R}_+ \}$.

$K \in \mathrm{GR}(\Delta \times^{\mathsf{RLD}} \uparrow\{x\})$ and thus $K \in \mathrm{GR} \bigsqcup \{ \Delta \times^{\mathsf{RLD}} \uparrow\{x\} \mid x \in \mathbb{R}_+ \}$. But $K \notin \mathrm{GR}(\Delta \times^{\mathsf{RLD}} \uparrow\mathbb{R}_+)$. □