Proof. If $\langle f\rangle X\notin\mu$ then $X\subseteq\langle f^{-1}\rangle\langle f\rangle X\notin\mu$ and so $X\notin\mu$. Thus $X\in\mu\wedge\langle f\rangle X\in\mu$ and consequently $X\cap\langle f\rangle X\in\mu$. □

We will say that x is *periodic* when $f^n(x)=x$ for some positive integer x. The least such n is called *the period* of x.

Let's define $x\sim y$ iff there exist $i,j\in\mathbb{N}$ such that $f^i(x)=f^j(y)$. Trivially it is an equivalence relation. If x and y are periodic, then $x\sim y$ iff exists $n\in\mathbb{N}$ such that $f^n(y)=x$.

Let $A=\{x\in I\mid x \text{ is periodic with period}>1\}$.

We will show that $A\notin\mu$. Let's assume $A\in\mu$.

Let a set $D\subseteq A$ contains (by the axiom of choice) exactly one element from each equivalence class of A defined by the relation $\sim$.

Let α be a function $A\to\mathbb{N}$ defined as follows. Let $x\in A$. Let y be the unique element of D such that $x\sim y$. Let $\alpha(x)$ be the least $n\in\mathbb{N}$ such that $f^n(y)=x$.

Let $B_0=\{x\in A\mid\alpha(x)\text{ is even}\}$ and $B_1=\{x\in A\mid\alpha(x)\text{ is odd}\}$.

Let $B_2=\{x\in A\mid\alpha(x)=0\}$.

Lemma 13.50. $B_0\cap\langle f\rangle B_0\subseteq B_2$.

Proof. If $x\in B_0\cap\langle f\rangle B_0$ then $f^n(y)=x$ for a minimal even n and $x=f(x')$ where $f^m(y')=x'$ for a minimal even m. Thus $f^n(y)=f(x')$ thus y and x' laying in the same equivalence class and thus $y=y'$. So we have $f^n(y)=f^{m+1}(y)$. Thus $n\leqslant m+1$ by minimality.

x' lies on an orbit and thus $x'=f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit; $f^m(y)=f^{-1}(x)$ and thus $x'=f^{n-1}(y)$ thus $n-1\geqslant m$ by minimality or $n=0$.

Thus $n=m+1$ what is impossible for even n and m. We have a contradiction what proves $B_0\cap\langle f\rangle B_0\subseteq\emptyset$.

Remained the case $n=0$, then $x=f^0(y)$ and thus $\alpha(x)=0$. □

Lemma 13.51. $B_1\cap\langle f\rangle B_1=\emptyset$.

Proof. Let $x\in B_1\cap\langle f\rangle B_1$. Then $f^n(y)=x$ for an odd n and $x=f(x')$ where $f^m(y')=x'$ for an odd m. Thus $f^n(y)=f(x')$ thus y and x' laying in the same equivalence class and thus $y=y'$. So we have $f^n(y)=f^{m+1}(y)$. Thus $n\leqslant m+1$ by minimality.

x' lies on an orbit and thus $x'=f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit;

$f^m(y)=f^{-1}(x)$ and thus $x'=f^{n-1}(y)$ thus $n-1\geqslant m$ by minimality ($n=0$ is impossible because n is odd).

Thus $n=m+1$ what is impossible for odd n and m. We have a contradiction what proves $B_1\cap\langle f\rangle B_1=\emptyset$. □

Lemma 13.52. $B_2\cap\langle f\rangle B_2=\emptyset$.

Proof. Let $x\in B_2\cap\langle f\rangle B_2$. Then $x=y$ and $x'=y$ where $x=f(x')$. Thus $x=f(x)$ and so $x\notin A$ what is impossible. □

Lemma 13.53. $A\notin\mu$.

Proof. Suppose $A\in\mu$.

Since $A\in\mu$ we have $B_0\in\mu$ or $B_1\in\mu$.

So either $B_0\cap\langle f\rangle B_0\subseteq B_2$ or $B_1\cap\langle f\rangle B_1\subseteq B_2$. As such by the lemma 13.49 we have $B_2\in\mu$. This is incompatible with $B_2\cap\langle f\rangle B_2=\emptyset$. So we got a contradiction. □

Let C be the set of points x which are not periodic but $f^n(x)$ is periodic for some positive n.

Lemma 13.54. $C\notin\mu$.

Proof. Let β be a function $C\to\mathbb{N}$ such that $\beta(x)$ is the least $n\in\mathbb{N}$ such that $f^n(x)$ is periodic.

Let $C_0=\{x\in C\mid\beta(x)\text{ is even}\}$ and $C_1=\{x\in C\mid\beta(x)\text{ is odd}\}$.

Obviously $C_j\cap\langle f\rangle C_j=\emptyset$ for $j=0,1$. Hence by lemma 13.49 we have $C_0,C_1\notin\mu$ and thus $C=C_0\cup C_1\notin\mu$. □