2. Let $f \in \mathrm{Mor}_{\mathbf{MonRld}_{\sqsubseteq,=}}(\mathcal{A};\mathcal{B})$, $g \in \mathrm{Mor}_{\mathbf{MonRld}_{\sqsubseteq,=}}(\mathcal{B};\mathcal{C})$. Then $\mathrm{dom}\, f \sqsubseteq \mathcal{A}$, $\mathrm{im}\, f = \mathcal{B}$, $\mathrm{dom}\, g \sqsubseteq \mathcal{B}$, $\mathrm{im}\, g = \mathcal{C}$. So $\mathrm{dom}(g \circ f) \sqsubseteq \mathcal{A}$, $\mathrm{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \mathrm{Mor}_{\mathbf{MonRld}_{\sqsubseteq,=}}(\mathcal{A};\mathcal{C})$.
3. Let $f \in \mathrm{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A};\mathcal{B})$, $g \in \mathrm{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{B};\mathcal{C})$. Then $\mathrm{dom}\, f = \mathcal{A}$, $\mathrm{im}\, f = \mathcal{B}$, $\mathrm{dom}\, g = \mathcal{B}$, $\mathrm{im}\, g = \mathcal{C}$. So $\mathrm{dom}(g \circ f) = \mathcal{A}$, $\mathrm{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \mathrm{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A};\mathcal{C})$. □

**Definition 13.44.** Let BijRld be the groupoid of all bijections of the category of reloid triples. Its objects are filters and its morphisms from a filter $\mathcal{A}$ to filter $\mathcal{B}$ are monovalued injective reloids $f$ such that $\mathrm{dom}\, f = \mathcal{A}$ and $\mathrm{im}\, f = \mathcal{B}$.

**Theorem 13.45.** Filters $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff $\mathrm{Mor}_{\mathbf{BijRld}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.

**Proof.**

**⇒.** Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic. Then there are sets $A \in \mathcal{A}$, $B \in \mathcal{B}$ and a bijective Set-morphism $F: A \to B$ such that $\langle F \rangle: \mathscr{P} A \cap \mathcal{A} \to \mathscr{P} B \cap \mathcal{B}$ is a bijection.

Obviously $f = (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}$ is monovalued and injective.

$\mathrm{im}\, f = \bigsqcap \{\uparrow^B \mathrm{im}\, G \mid G \in (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}\} = \bigsqcap \{\uparrow^B \mathrm{im}(H \cap F|_X) \mid H \in (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}, X \in \mathcal{A}\} = \bigsqcap \{\uparrow^B \mathrm{im}\ F|_P \mid P \in \mathcal{A}\} = \bigsqcap \{\uparrow^B \langle F \rangle P \mid P \in \mathcal{A}\} = \bigsqcap \{\uparrow^B \langle F \rangle P \mid P \in \mathscr{P} A \cap \mathcal{A}\} = \bigsqcap \langle \uparrow^B \rangle (\mathscr{P} B \cap \mathcal{B}) = \bigsqcap \langle \uparrow^B \rangle \mathcal{B} = \mathcal{B}$.

Thus $\mathrm{dom}\, f = \mathcal{A}$ and $\mathrm{im}\, f = \mathcal{B}$.

**⇐.** Let $f$ be a monovalued injective reloid such that $\mathrm{dom}\, f = \mathcal{A}$ and $\mathrm{im}\, f = \mathcal{B}$. Then there exist a function $F'$ and an injective binary relation $F''$ such that $F', F'' \in \mathrm{GR}\, f$. Thus $F = F' \cap F''$ is an injection such that $F \in \mathrm{GR}\, f$. The function $F$ is a bijection from $A = \mathrm{dom}\, F$ to $B = \mathrm{im}\, F$. The function $\langle F \rangle$ is an injection on $\mathscr{P} A \cap \mathcal{A}$ (and moreover on $\mathscr{P} A$). It's simple to show that $\forall X \in \mathscr{P} A \cap \mathcal{A}: \langle F \rangle X \in \mathscr{P} B \cap \mathcal{B}$ and similarly $\forall Y \in \mathscr{P} B \cap \mathcal{B}: \langle F \rangle^{-1} Y = \langle F^{-1} \rangle Y \in \mathscr{P} A \cap \mathcal{A}$. Thus $\langle F \rangle|_{\mathscr{P} A \cap \mathcal{A}}$ is a bijection $\mathscr{P} A \cap \mathcal{A} \to \mathscr{P} B \cap \mathcal{B}$. So filters $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. □

**Proposition 13.46.** $(\geqslant_1) = (\sqsupseteq) \circ (\geqslant_2)$ (when we limit to small filters).

**Proof.** $\mathcal{A} \geqslant_1 \mathcal{B}$ iff exists a function $f: \mathrm{Base}(\mathcal{A}) \to \mathrm{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. But $\mathcal{B} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$ is equivalent to $\exists \mathcal{B}' \in \mathfrak{F}: (\mathcal{B}' \sqsupseteq \mathcal{B} \wedge \mathcal{B}' = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A})$. So $\mathcal{A} \geqslant_1 \mathcal{B}$ is equivalent to existence of $\mathcal{B}' \in \mathfrak{F}$ such that $\mathcal{B}' \sqsupseteq \mathcal{B}$ and existence of a function $f: \mathrm{Base}(\mathcal{A}) \to \mathrm{Base}(\mathcal{B})$ such that $\mathcal{B}' = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. That is equivalent to $\mathcal{A}((\sqsupseteq) \circ (\geqslant_2))\mathcal{B}$. □

**Proposition 13.47.** If $a$ and $b$ are ultrafilters then $b \geqslant_1 a \Leftrightarrow b \geqslant_2 a$.

**Proof.** We need to prove only $b \geqslant_1 a \Rightarrow b \geqslant_2 a$. If $b \geqslant_1 a$ then there exists a monovalued reloid $f: \mathrm{Base}(b) \to \mathrm{Base}(a)$ such that $\mathrm{dom}\, f = b$ and $\mathrm{im}\, f \sqsupseteq a$. Then $\mathrm{im}\, f = \mathrm{im}\, (\mathsf{FCD}) f \in \left\{0^{\mathfrak{F}(\mathrm{Base}(a))}\right\} \cup \mathrm{atoms}^{\mathfrak{F}(\mathrm{Base}(a))}$ because $(\mathsf{FCD}) f$ is a monovalued funcoid. So $\mathrm{im}\, f = a$ (taken into account $a \neq 0^{\mathfrak{F}(\mathrm{Base}(a))}$) and thus $b \geqslant_2 a$. □

**Corollary 13.48.** For atomic filters $\geqslant_1$ is the same as $\geqslant_2$.

Thus I will write simply $\geqslant$ for atomic filters.

### 13.2.1 Existence of no more than one monovalued injective reloid for a given pair of ultrafilters

#### 13.2.1.1 The lemmas

The lemmas in this section were provided to me by Robert Martin Solovay in [36]. They are based on Wistar Comfort's work.

In this section we will assume $\mu$ is an ultrafilter on a set $I$ and function $f: I \to I$ has the property $X \in \mu \Leftrightarrow \langle f^{-1} \rangle X \in \mu$.

**Lemma 13.49.** If $X \in \mu$ then $X \cap \langle f \rangle X \in \mu$.