3. $\mathrm{Mor}_{\mathrm{MonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{A};\mathcal{B})\neq\emptyset$.
4. $\mathrm{Mor}_{\mathrm{MonRld}_{\sqsubseteq,=}}(\mathcal{A};\mathcal{B})\neq\emptyset$.
5. $\mathrm{Mor}_{\mathrm{CoMonRld}_{=,\sqsupseteq}}(\mathcal{A};\mathcal{B})\neq\emptyset$.
6. $\mathrm{Mor}_{\mathrm{CoMonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{A};\mathcal{B})\neq\emptyset$.
7. $\mathrm{Mor}_{\mathrm{CoMonRld}_{\sqsubseteq,=}}(\mathcal{A};\mathcal{B})\neq\emptyset$.

Proof.

(1)⇒(2). There exists a Set-morphism $f\colon \mathrm{Base}(\mathcal{A})\to\mathrm{Base}(\mathcal{B})$ such that $\mathcal{B}\sqsubseteq\langle\uparrow^{\mathsf{FCD}}f\rangle\mathcal{A}$. We have

$$\mathrm{dom}\,(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}=\mathcal{A}\sqcap 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))}=\mathcal{A}$$

and

$$\mathrm{im}\,(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}=\mathrm{im}\,(\mathsf{FCD})(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}=\mathrm{im}\,(\uparrow^{\mathsf{FCD}}f)|_{\mathcal{A}}=\langle\uparrow^{\mathsf{FCD}}f\rangle\mathcal{A}\sqsupseteq\mathcal{B}.$$

Thus $(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}$ is a monovalued reloid such that $\mathrm{dom}\,(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}=\mathcal{A}$ and $\mathrm{im}\,(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}\sqsupseteq\mathcal{B}$.

(2)⇒(3), (4)⇒(3), (5)⇒(6), (7)⇒(6). Obvious.

(3)⇒(1). We have $\mathcal{B}\sqsubseteq\langle(\mathsf{FCD})f\rangle\mathcal{A}$ for a monovalued reloid $f\in\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))$. Then there exists a Set-morphism $F\colon\mathrm{Base}(\mathcal{A})\to\mathrm{Base}(\mathcal{B})$ such that $\mathcal{B}\sqsubseteq\langle\uparrow^{\mathsf{FCD}}F\rangle\mathcal{A}$ that is $\mathcal{A}\geqslant_1\mathcal{B}$.

(6)⇒(7). $\mathrm{dom}\,f|_{\mathcal{B}}=\mathcal{B}$ and $\mathrm{im}\,f|_{\mathcal{B}}\sqsubseteq\mathcal{A}$.

(2)⇔(5), (3)⇔(6), (4)⇔(7). By duality. □

Theorem 13.42. For every filters $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A}\geqslant_2\mathcal{B}$.
2. $\mathrm{Mor}_{\mathrm{MonRld}_{=,=}}(\mathcal{A};\mathcal{B})\neq\emptyset$.
3. $\mathrm{Mor}_{\mathrm{CoMonRld}_{=,=}}(\mathcal{A};\mathcal{B})\neq\emptyset$.

Proof.

(1)⇒(2). Let $\mathcal{A}\geqslant_2\mathcal{B}$ that is $\mathcal{B}=\langle\uparrow^{\mathsf{FCD}}f\rangle\mathcal{A}$ for some Set-morphism $f\colon\mathrm{Base}(\mathcal{A})\to\mathrm{Base}(\mathcal{B})$. Then $\mathrm{dom}\,(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}=\mathcal{A}$ and $\mathrm{im}\,(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}=\mathrm{im}\,(\mathsf{FCD})(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}=\mathrm{im}\,(\uparrow^{\mathsf{FCD}}f)|_{\mathcal{A}}=\langle\uparrow^{\mathsf{FCD}}f\rangle\mathcal{A}=\mathcal{B}$. So $(\uparrow^{\mathsf{RLD}}f)|_{\mathcal{A}}$ is a sought for reloid.

(2)⇒(1). By corollary 13.78 below, there exists a Set-morphism $F\colon\mathrm{Base}(\mathcal{A})\to\mathrm{Base}(\mathcal{B})$ such that $f=(\uparrow^{\mathsf{RLD}}F)|_{\mathcal{A}}$. Thus $\langle\uparrow^{\mathsf{FCD}}F\rangle\mathcal{A}=\mathrm{im}(\uparrow^{\mathsf{FCD}}F)|_{\mathcal{A}}=\mathrm{im}\,(\mathsf{FCD})(\uparrow^{\mathsf{RLD}}F)|_{\mathcal{A}}=\mathrm{im}\,(\mathsf{FCD})f=\mathrm{im}\,f=\mathcal{B}$. Thus $\mathcal{A}\geqslant_2\mathcal{B}$ is testified by the morphism F.

(2)⇔(3). By duality. □

Theorem 13.43. The following are categories (with reloid composition):

1. $\mathrm{MonRld}_{\sqsubseteq,\sqsupseteq}$;
2. $\mathrm{MonRld}_{\sqsubseteq,=}$;
3. $\mathrm{MonRld}_{=,=}$.
4. $\mathrm{CoMonRld}_{\sqsubseteq,\sqsupseteq}$;
5. $\mathrm{CoMonRld}_{\sqsubseteq,=}$;
6. $\mathrm{CoMonRld}_{=,=}$.

Proof. We will prove only the first three. The rest follow from duality.[TODO: Check duality.] We need to prove only that composition of morphisms is a morphism, because associativity and existence of identity morphism are evident. We have:

1. Let $f\in\mathrm{Mor}_{\mathrm{MonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{A};\mathcal{B})$, $g\in\mathrm{Mor}_{\mathrm{MonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{B};\mathcal{C})$. Then $\mathrm{dom}\,f\sqsubseteq\mathcal{A}$, $\mathrm{im}\,f\sqsupseteq\mathcal{B}$, $\mathrm{dom}\,g\sqsubseteq\mathcal{B}$, $\mathrm{im}\,g\sqsupseteq\mathcal{C}$. So $\mathrm{dom}(g\circ f)\sqsubseteq\mathcal{A}$, $\mathrm{im}(g\circ f)\sqsupseteq\mathcal{C}$ that is $g\circ f\in\mathrm{Mor}_{\mathrm{MonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{A};\mathcal{C})$.