If $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ our statement follows from the last lemma.

Now assume without loss of generality $\operatorname{card}(X \setminus A) < \operatorname{card}(Y \setminus B)$.

$\operatorname{card} B = \operatorname{card} Y$ because $\operatorname{card}(Y \setminus B) < \operatorname{card} Y$.

It is easy to show that there exists $B' \supset B$ such that $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B')$ and $\operatorname{card} B' = \operatorname{card} B$.

We will find a bijection g from B to B' which witnesses direct isomorphism of v to v itself. Then the composition $g \circ f$ witnesses a direct isomorphism of $u \div A$ and $v \div B'$ and by the lemma u and v are directly isomorphic.

Let $D = B' \setminus B$. We have $D \notin v$.

There exists a set $E \subseteq B$ such that $\operatorname{card} E \geqslant \operatorname{card} D$ and $E \notin v$.

We have $\operatorname{card} E = \operatorname{card}(D \cup E)$ and thus there exists a bijection $h: E \to D \cup E$.

Let

$$g(x) = \begin{cases} x & \text{if } x \in B \setminus E; \\ h(x) & \text{if } x \in E. \end{cases}$$

$g|_{B \setminus E}$ and $g|_E$ are bijections.

$\operatorname{im}(g|_{B \setminus E}) = B \setminus E$; $\operatorname{im}(g|_E) = \operatorname{im} h = D \cup E$;

$$(D \cup E) \cap (B \setminus E) = (D \cap (B \setminus E)) \cup (E \cap (B \setminus E)) = \emptyset \cup \emptyset = \emptyset.$$

Thus g is a bijection from B to $(B \setminus E) \cup (D \cup E) = B \cup D = B'$.

To finish the proof it's enough to show that $\langle g \rangle v = v$. Indeed it follows from $B \setminus E \in v$. □

Proposition 13.36.

1. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $\mathcal{A} \geqslant_2 \mathcal{B}$ iff $\mathcal{A} \div A \geqslant_2 \mathcal{B} \div B$.
2. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $\mathcal{A} \geqslant_1 \mathcal{B}$ iff $\mathcal{A} \div A \geqslant_1 \mathcal{B} \div B$.

Proof.

1. $\mathcal{A} \geqslant_2 \mathcal{B}$ iff there exist a bijective Set-morphism f such that $\mathcal{B} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$.
2. $\mathcal{A} \geqslant_1 \mathcal{B}$ iff there exist a bijective Set-morphism f such that $\mathcal{B} \subseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$. □

Proposition 13.37. For ultrafilters $\geqslant_2$ is the same as Rudin-Keisler ordering (as defined in [37]).

Proof. $x \geqslant_2 y$ iff there exist sets $A \in x$ and $B \in y$ a bijective Set-morphism $f: X \to Y$ such that $y \div B = \{C \in \mathscr{P} Y \mid \langle f^{-1} \rangle C \in x \div A\}$ that is when $C \in y \div B \Leftrightarrow \langle f^{-1} \rangle C \in x \div A$ what is equivalent to $C \in y \Leftrightarrow \langle f^{-1} \rangle C \in x$ what is the definition of Rudin-Keisler ordering. □

Remark 13.38. The relation of being isomorphic for ultrafilters is traditionally called *Rudin-Keisler equivalence*.

Obvious 13.39. $(\geqslant_1) \supseteq (\geqslant_2)$.

Definition 13.40. Let Q and R be binary relations on the set of filters. I will denote $\mathrm{MonRld}_{Q,R}$ the directed multigraph with objects being filters and morphisms such monovalued reloids f that $(\operatorname{dom} f)\, Q\, \mathcal{A}$ and $(\operatorname{im} f)\, R\, \mathcal{B}$.

I will also denote $\mathrm{CoMonRld}_{Q,R}$ the directed multigraph with objects being filters and morphisms such injective reloids f that $(\operatorname{im} f)\, Q\, \mathcal{A}$ and $(\operatorname{dom} f)\, R\, \mathcal{B}$. These are essentially the duals.

Some of these directed multigraphs are categories with reloid composition (see below). By abuse of notation I will denote these categories the same as these directed multigraphs.

Theorem 13.41. For every filters $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \geqslant_1 \mathcal{B}$.
2. $\mathrm{Mor}_{\mathrm{MonRld}_{=,\sqsupseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.