Proof. First let's prove it is a category. Let $f: \mathcal{A} \rightarrow \mathcal{B}$ and $g: \mathcal{B} \rightarrow \mathcal{C}$ be morphisms of FuncBij. Then $f: \mathrm{Base}(\mathcal{A}) \rightarrow \mathrm{Base}(\mathcal{B})$ and $g: \mathrm{Base}(\mathcal{A}) \rightarrow \mathrm{Base}(\mathcal{C})$ are bijections and $\mathcal{B} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$ and $\mathcal{C} = \langle \uparrow^{\mathsf{FCD}} g \rangle \mathcal{B}$. Thus $g \circ f: \mathrm{Base}(\mathcal{A}) \rightarrow \mathrm{Base}(\mathcal{C})$ is a bijection and $\mathcal{C} = \langle \uparrow^{\mathsf{FCD}} (g \circ f) \rangle \mathcal{A}$. Thus $g \circ f$ is a morphism of FuncBij. $\mathrm{id}_{\mathrm{Base}(\mathcal{A})}$ is the identity morphism of FuncBij for every filter $\mathcal{A}$. Thus it is a category.

It remains to prove only that every morphism $f \in \mathrm{Mor}_{\mathrm{FuncBij}}(\mathcal{A}; \mathcal{B})$ has a reverse (for every filters $\mathcal{A}$, $\mathcal{B}$). We have f is a bijection $\mathrm{Base}(\mathcal{A}) \rightarrow \mathrm{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P}\mathrm{Base}(\mathcal{A})$

$$\langle f \rangle C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A}.$$

Then $f^{-1}: \mathrm{Base}(\mathcal{B}) \rightarrow \mathrm{Base}(\mathcal{A})$ is a bijection such that for every $C \in \mathscr{P}\mathrm{Base}(\mathcal{B})$

$$\langle f^{-1} \rangle C \in \mathcal{A} \Leftrightarrow C \in \mathcal{B}.$$

Thus $f^{-1} \in \mathrm{Mor}_{\mathrm{FuncBij}}(\mathcal{B}; \mathcal{A})$. □

Corollary 13.29. Being directly isomorphic is an equivalence relation.

Rudin-Keisler order of ultrafilters is considered in such a book as [37].

Obvious 13.30. For the case of ultrafilters being directly isomorphic is the same as being Rudin-Keisler equivalent.

Definition 13.31. A filter $\mathcal{A}$ is *isomorphic* to a filter $\mathcal{B}$ iff there exist sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$.

Obvious 13.32. Equivalent filters are isomorphic.

Theorem 13.33. Being isomorphic (for small filters) is an equivalence relation.

Proof.

Reflexivity. Because every filter is directly isomorphic to itself.

Symmetry. If filter $\mathcal{A}$ is isomorphic to $\mathcal{B}$ then there exist sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$ and thus $\mathcal{B} \div B$ is directly isomorphic to $\mathcal{A} \div A$. So $\mathcal{B}$ is isomorphic to $\mathcal{A}$.

Transitivity. Let $\mathcal{A}$ be isomorphic to $\mathcal{B}$ and $\mathcal{B}$ be isomorphic to $\mathcal{C}$. Then exist $A \in \mathcal{A}$, $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$, $C \in \mathcal{C}$ such that there are bijections $f: A \rightarrow B_1$ and $g: B_2 \rightarrow C$ such that

$$\forall X \in \mathscr{P}A: (X \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle X \in \mathcal{A}) \quad \text{and} \quad \forall X \in \mathscr{P}B_2: (X \in \mathcal{A} \Leftrightarrow \langle f \rangle X \in \mathcal{B}).$$

Also $\forall X \in \mathscr{P}B_2: (X \in \mathcal{B} \Leftrightarrow \langle g \rangle X \in \mathcal{C})$.

So $g \circ f$ is a bijection from $\langle f^{-1} \rangle (B_1 \cap B_2) \in \mathcal{A}$ to $\langle g \rangle (B_1 \cap B_2) \in \mathcal{C}$ such that

$$X \in \mathcal{A} \Leftrightarrow \langle f \rangle X \in \mathcal{B} \Leftrightarrow \langle g \rangle \langle f \rangle X \in \mathcal{C} \Leftrightarrow \langle g \circ f \rangle X \in \mathcal{C}.$$

Thus $g \circ f$ establishes a bijection which proves that $\mathcal{A}$ is isomorphic to $\mathcal{C}$. □

Lemma 13.34. Let $\mathrm{card}\, X = \mathrm{card}\, Y$, u be an ultrafilter on X and v be an ultrafilter on Y; let $A \in u$ and $B \in v$. Let $u \div A$ and $v \div B$ be directly isomorphic. Then if $\mathrm{card}(X \setminus A) = \mathrm{card}(Y \setminus B)$ we have u and v directly isomorphic.

Proof. Arbitrary extend the bijection witnessing being directly isomorphic to the sets $X \setminus A$ and $Y \setminus B$. □

Theorem 13.35. If $\mathrm{card}\, X = \mathrm{card}\, Y$ then being isomorphic and being directly isomorphic are the same for ultrafilters u on X and v on Y.

Proof. That if two filters are isomorphic then they are directly isomorphic is obvious.

Let ultrafilters u and v be isomorphic that is there is a bijection $f: A \rightarrow B$ where $A \in u$, $B \in v$ witnessing isomorphism of u and v.

If one of the filters u or v is a trivial ultrafilter then the other is also a trivial ultrafilter and as it is easy to show they are directly isomorphic. So we can assume u and v are not trivial ultrafilters.