Chapter 13
Orderings of filters in terms of reloids

Whilst the other chapters of this book use filters to research funcoids and reloids, here the opposite thing is discussed, the theory of reloids is used to describe properties of filters.

In this chapter the word *filter* is used to denote a filter on a set (not on an arbitrary poset) only.

13.1 Equivalent filters

Definition 13.1. Two filters $\mathcal{A}$ and $\mathcal{B}$ (with possibly different base sets) are *equivalent* ($\mathcal{A}\sim\mathcal{B}$) iff there exists a set X such that $X\in\mathcal{A}$ and $X\in\mathcal{B}$ and $\mathscr{P}X\cap\mathcal{A}=\mathscr{P}X\cap\mathcal{B}$.

Proposition 13.2. If two filters with the same base are equivalent they are equal.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two filters and $\mathscr{P}X\cap\mathcal{A}=\mathscr{P}X\cap\mathcal{B}$ for some set X such that $X\in\mathcal{A}$ and $X\in\mathcal{B}$, and $\operatorname{Base}(\mathcal{A})=\operatorname{Base}(\mathcal{B})$. Then $\mathcal{A}=(\mathscr{P}X\cap\mathcal{A})\cup\{Y\in\mathscr{P}\operatorname{Base}(\mathcal{A})\mid Y\supseteq X\}=(\mathscr{P}X\cap\mathcal{B})\cup\{Y\in\mathscr{P}\operatorname{Base}(\mathcal{B})\mid Y\supseteq X\}=\mathcal{B}$. □

Proposition 13.3. $\sim$ restricted to small filters is an equivalence relation.

Proof.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $\mathcal{A}\sim\mathcal{B}$ and $\mathcal{B}\sim\mathcal{C}$ for some small filters $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Then there exist a set X such that $X\in\mathcal{A}$ and $X\in\mathcal{B}$ and $\mathscr{P}X\cap\mathcal{A}=\mathscr{P}X\cap\mathcal{B}$ and a set Y such that $Y\in\mathcal{B}$ and $Y\in\mathcal{C}$ and $\mathscr{P}Y\cap\mathcal{B}=\mathscr{P}Y\cap\mathcal{C}$. So $X\cap Y\in\mathcal{A}$ because

$$\mathscr{P}Y\cap\mathscr{P}X\cap\mathcal{A}=\mathscr{P}Y\cap\mathscr{P}X\cap\mathcal{B}=\mathscr{P}(X\cap Y)\cap\mathcal{B}\supseteq\{X\cap Y\}\cap\mathcal{B}\ni X\cap Y.$$

Similarly we have $X\cap Y\in\mathcal{C}$. Finally $\mathscr{P}(X\cap Y)\cap\mathcal{A}=\mathscr{P}Y\cap\mathscr{P}X\cap\mathcal{A}=\mathscr{P}Y\cap\mathscr{P}X\cap\mathcal{B}=\mathscr{P}X\cap\mathscr{P}Y\cap\mathcal{B}=\mathscr{P}X\cap\mathscr{P}Y\cap\mathcal{C}=\mathscr{P}(X\cap Y)\cap\mathcal{C}$. □

Definition 13.4. The *rebase* $\mathcal{A}\div A$ for a filter $\mathcal{A}$ and a set A (base) such that $\exists X\in\mathcal{A}: X\subseteq A$ is defined by the formula

$$\mathcal{A}\div A=\{X\in\mathscr{P}A\mid\exists Y\in\mathcal{A}: Y\subseteq X\}.$$

Proposition 13.5. If $\exists X\in\mathcal{A}: X\subseteq A$ then:

1. $\mathcal{A}\div A$ is a filter on A;
2. $\mathcal{A}\div A\sim\mathcal{A}$.

Proof.

1. We need to prove that $\{X\in\mathscr{P}A\mid\exists Y\in\mathcal{A}: Y\subseteq X\}$ is a filter on A. That it is an upper set is obvious. It is non-empty because $\exists Y\in\mathcal{A}: Y\subseteq A$ and thus $A\in\{X\in\mathscr{P}A\mid\exists Y\in\mathcal{A}: Y\subseteq X\}$. Let $P,Q\in\{X\in\mathscr{P}A\mid\exists Y\in\mathcal{A}: Y\subseteq X\}$. Then $P,Q\subseteq A$ and $\exists P'\in\mathcal{A}: P'\subseteq P$ and $\exists Q'\in\mathcal{A}: Q'\subseteq Q$. So $P\cap Q\subseteq A$ and $P'\cap Q'\subseteq P\cap Q$. Thus $P\cap Q\in\{X\in\mathscr{P}A\mid\exists Y\in\mathcal{A}: Y\subseteq X\}$.