Proof. $\uparrow^{\mathrm{Ob}\,\mu} A$ is connected regarding a reloid f iff A is connected regarding every $F \in \mathrm{xyGR}\, f$ that is when (taken into account that connectedness for $\uparrow^{\mathsf{RLD}} F$ is the same as connectedness of $\uparrow^{\mathsf{FCD}} F$)

$$\begin{array}{rl} \forall F \in \mathrm{xyGR}\, f \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\} \colon (\mathcal{X} \sqcup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \mathcal{X}\ [\uparrow^{\mathsf{FCD}} F]\ \mathcal{Y}) & \Leftrightarrow \\ \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\} \forall F \in \mathrm{xyGR}\, f \colon (\mathcal{X} \sqcup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \mathcal{X}\ [\uparrow^{\mathsf{FCD}} F]\ \mathcal{Y}) & \Leftrightarrow \\ \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\} \colon (\mathcal{X} \sqcup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \forall F \in \mathrm{xyGR}\, f \colon \mathcal{X}\ [\uparrow^{\mathsf{FCD}} F]\ \mathcal{Y}) & \Leftrightarrow \\ \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\} \colon (\mathcal{X} \sqcup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \mathcal{X}\ [(\mathsf{FCD}) f]\ \mathcal{Y}) & \end{array}$$

that is when the set $\uparrow^{\mathrm{Ob}\, f} A$ is connected regarding the funcoid $(\mathsf{FCD}) f$. □

Conjecture 11.34. A set A is connected regarding an endofuncoid μ iff for every $a, b \in A$ there exists a totally ordered set $P \subseteq A$ such that $\min P = a$, $\max P = b$ and

$$\forall q \in P \setminus \{b\} \colon \{x \in P \mid x \leqslant q\}\ [\mu]^*\ \{x \in P \mid x > q\}.$$

Weaker condition:

$$\forall q \in P \setminus \{b\} \colon \{x \in P \mid x \leqslant q\}\ [\mu]^*\ \{x \in P \mid x > q\} \vee \forall q \in P \setminus \{a\} \colon \{x \in P \mid x < q\}\ [\mu]^*\ \{x \in P \mid x \geqslant q\}.$$

11.5 Algebraic properties of S and S^*

Theorem 11.35. $S^*(S^*(f)) = S^*(f)$ for every endoreloid f.

Proof. $S^*(S^*(f)) = \bigsqcap \{\uparrow^{\mathsf{RLD}} S(R) \mid R \in \mathrm{xyGR}\ S^*(f)\} \sqsubseteq \bigsqcap \{\uparrow^{\mathsf{RLD}} S(R) \mid R \in \{S(F) \mid F \in \mathrm{xyGR}\, f\}\} = \bigsqcap \{\uparrow^{\mathsf{RLD}} S(S(R)) \mid R \in \mathrm{xyGR}\, f\} = \bigsqcap \{\uparrow^{\mathsf{RLD}} S(R) \mid R \in \mathrm{xyGR}\, f\} = S^*(f)$.

So $S^*(S^*(f)) \sqsubseteq S^*(f)$. That $S^*(S^*(f)) \sqsupseteq S^*(f)$ is obvious. □

Corollary 11.36. $S^*(S(f)) = S(S^*(f)) = S^*(f)$ for every endoreloid f.

Proof. Obviously $S^*(S(f)) \sqsupseteq S^*(f)$ and $S(S^*(f)) \sqsupseteq S^*(f)$.

But $S^*(S(f)) \sqsubseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \sqsubseteq S^*(S^*(f)) = S^*(f)$. □

Conjecture 11.37. $S(S(f)) = S(f)$ for

1. every endoreloid f;
2. every endofuncoid f.

Conjecture 11.38. For every endoreloid f

1. $S(f) \circ S(f) = S(f)$;
2. $S^*(f) \circ S^*(f) = S^*(f)$;
3. $S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$.

Conjecture 11.39. $S(f) \circ S(f) = S(f)$ for every endofuncoid f.