Proof.

$$\begin{aligned} S(\mu)\circ S(\mu) &= \mu^0 \sqcup S(\mu)\sqcup \mu\circ S(\mu)\sqcup \mu^2\circ S(\mu)\sqcup \dots \\ &= (\mu^0\sqcup\mu^1\sqcup\mu^2\sqcup\dots)\sqcup(\mu^1\sqcup\mu^2\sqcup\mu^3\sqcup\dots)\sqcup(\mu^2\sqcup\mu^3\sqcup\mu^4\sqcup\dots) \\ &= \mu^0\sqcup\mu^1\sqcup\mu^2\sqcup\dots \\ &= S(\mu). \end{aligned}$$

□

11.3 Connectedness regarding binary relations

Before going to research connectedness for funcoids and reloids we will excurse into the basic special case of connectedness regarding binary relations on a set $\mho$.

Definition 11.8. A set A is called *(strongly) connected* regarding a binary relation μ when

$$\forall X\in\mathscr{P}(\operatorname{dom}\mu)\setminus\{\emptyset\}, Y\in\mathscr{P}(\operatorname{im}\mu)\setminus\{\emptyset\}: (X\cup Y = A\Rightarrow X\,[\mu]\,Y).$$

Let $\mho$ be a set.

Definition 11.9. *Path* between two elements $a,b\in\mho$ in a set $A\subseteq\mho$ through binary relation μ is the finite sequence $x_0\dots x_n$ where $x_0=a$, $x_n=b$ for $n\in\mathbb{N}$ and $x_i\,(\mu\cap A\times A)\,x_{i+1}$ for every $i=0,\dots,n-1$. n is called path *length*.

Proposition 11.10. There exists path between every element $a\in\mho$ and that element itself.

Proof. It is the path consisting of one vertex (of length 0). □

Proposition 11.11. There is a path from element a to element b in a set A through a binary relation μ iff $a\,(S(\mu\cap A\times A))\,b$ (that is $(a,b)\in S(\mu\cap A\times A)$).

Proof.

⇒. If a path from a to b exists, then $\{b\}\subseteq\langle(\mu\cap A\times A)^n\rangle\{a\}$ where n is the path length. Consequently $\{b\}\subseteq\langle S(\mu\cap A\times A)\rangle\{a\}$; $a\,(S(\mu\cap A\times A))\,b$.

⇐. If $a\,(S(\mu\cap A\times A))\,b$ then there exists $n\in\mathbb{N}$ such that $a\,(\mu\cap A\times A)^n\,b$. By definition of composition of binary relations this means that there exist finite sequence $x_0\dots x_n$ where $x_0=a$, $x_n=b$ for $n\in\mathbb{N}$ and $x_i\,(\mu\cap A\times A)\,x_{i+1}$ for every $i=0,\dots,n-1$. That is there is a path from a to b. □

Theorem 11.12. The following statements are equivalent for a binary relation μ and a set A:

1. For every $a,b\in A$ there is a path between a and b in A through μ.
2. $S(\mu\cap(A\times A))\supseteq A\times A$.
3. $S(\mu\cap(A\times A))=A\times A$.
4. A is connected regarding μ.

Proof.

(1)⇒(2). Let for every a, $b\in A$ there is a path between a and b in A through μ. Then $a\,(S(\mu\cap A\times A))\,b$ for every $a,b\in A$. It is possible only when $S(\mu\cap(A\times A))\supseteq A\times A$.

(3)⇒(1). For every two vertices a and b we have $a\,(S(\mu\cap A\times A))\,b$. So (by the previous theorem) for every two vertices a and b there exists a path from a to b.

(3)⇒(4). Suppose $\neg(X\,[\mu\cap(A\times A)]\,Y)$ for some $X,Y\in\mathscr{P}\mho\setminus\{\emptyset\}$ such that $X\cup Y=A$. Then by a lemma $\neg(X\,[(\mu\cap(A\times A))^n]\,Y)$ for every $n\in\mathbb{N}$. Consequently $\neg(X\,[S(\mu\cap(A\times A))]\,Y)$. So $S(\mu\cap(A\times A))\neq A\times A$.